A FAMILY OF POISSON NON-COMPACT SYMMETRIC SPACES

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ABSTRACT. We study Poisson symmetric spaces of group type with Cartan subalgebra “adapted" to the Lie cobracket.

INTRODUCTION

Let A be a Poisson-Lie group and T a Lie subgroup of A. The homogeneous space A/T endowed with a Poisson structure is a *Poisson homogeneous space* if the action $A \times A/T \to A/T$ is a morphism of Poisson manifolds. Poisson homogeneous spaces, after the seminal paper [D3], have been studied by several authors, see [EL, FL, L1, KRR, K, KoS]; and also in connection with the quantum dynamical Yang-Baxter equation, see [EE, EEM, KS, L2] and references therein.

If A/T carries a Poisson structure such that the natural projection $A \to A/T$ is a morphism of Poisson manifolds, then A/T is a Poisson homogeneous space, and in this case is said to be *of group type*. Assume that A and T are connected. Let $\mathfrak{a}$, $\mathfrak{t}$ denote the Lie algebras of A, T and let $\delta : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$ be the Lie cobracket inherited from the Poisson structure of A [D1], see also [KoS, Th. 3.3.1]. Then the following conditions are equivalent– see [S, KRR]:

(i) A/T is a Poisson homogeneous space of group type;

(ii) $\{\mu \in \mathfrak{a}^* : \mu(\mathfrak{t}) = 0\}$ is a subalgebra in $\mathfrak{a}^*$;

(iii) $\mathfrak{t}$ is a coideal of $\mathfrak{a}$, i.e. $\delta(\mathfrak{t}) \subset \mathfrak{t} \otimes \mathfrak{a} + \mathfrak{a} \otimes \mathfrak{t}$.

In this paper, we study Poisson non-compact symmetric spaces of group type. That is, we assume the following setting:

- $A = G_0$ is a non-compact absolutely simple real Lie group with finite center and Lie algebra $\mathfrak{g}_0$; we fix a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$;
- the Poisson-Lie group structure on G_0 corresponds to an almost factorizable Lie bialgebra structure on $\mathfrak{g}_0$;
- $T = K_0$ is a connected Lie subgroup with Lie algebra $\mathfrak{k}_0$ (in other words, K_0 is a maximal compact subgroup of G_0).

We note that the symmetric space G_0/K_0 always has a structure of Poisson homogeneous space, see subsection 1.5. However, whether this Poisson homogeneous structure is of group type is not evident.

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Almost factorizable Lie bialgebra structures on $\mathfrak{g}_0$ were classified in [AJ], starting from the analogous classification in the complex case [BD]. In particular, to each almost factorizable Lie bialgebra structure δ on $\mathfrak{g}_0$ corresponds a unique Cartan subalgebra $\mathfrak{h}$ of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$, a unique system of simple roots Δ in the set of roots $\Phi(\mathfrak{g},\mathfrak{h})$, a unique continuous parameter $\lambda \in \mathfrak{h}^{\otimes 2}$ and a unique Belavin-Drinfeld triple (Γ_1, Γ_2, T). Here Γ_1, Γ_2 are subsets of Δ and $T : \Gamma_1 \to \Gamma_2$ (see subsection 1.4).

On the other hand, all maximal compact Lie subgroups of G_0 are conjugated, and they actually arise as the fixed point set of the Chevalley involution corresponding to some Cartan subalgebra of $\mathfrak{g}$ and some system of simple roots (see subsection 1.2). Let δ an almost factorizable Lie bialgebra structure on $\mathfrak{g}_0$, and $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra and Δ the set of simple roots determined by δ. Let ω be the Chevalley involution that arises from $\mathfrak{h}$ and Δ. We say that K_0 is *adapted to* δ if the Lie algebra of K_0 is the fixed point set of ω.

Now, let $\mu : \Delta \to \Delta$ be an automorphism of the Dynkin diagram of order 1 or 2. Let J be any subset of the set Δ^μ of simple roots fixed by μ. With this data we can define unique conjugate-linear Lie algebra involutions ς_μ, $\omega_{\mu,J}$ of $\mathfrak{g}$, see subsection 1.2 again. It is a well known result that if $\mathfrak{g}_0$ is an absolutely simple real Lie algebra, then it is the set of fixed points of $\mathfrak{g}$, the complexification of $\mathfrak{g}_0$, by $\sigma : \mathfrak{g} \to \mathfrak{g}$ a conjugate linear involution, where σ is ς_μ or $\omega_{\mu,J}$ for some μ and J (if applies). We denote $\varsigma = \varsigma_{\mathrm{id}}$, $\omega_J = \omega_{\mathrm{id},J}$ and $\omega = \omega_{\mathrm{id},\Delta}$ (the Chevalley involution). Here is the main result of the paper.

Theorem 1. *Let $(\mathfrak{g}_0, \delta)$ be an almost factorizable absolutely simple real Lie bialgebra, let σ be the conjugate-linear involution of $\mathfrak{g}$ such that $\mathfrak{g}_0 = \mathfrak{g}^\sigma$, and let K_0 be the maximal compact Lie subgroup of G_0 adapted to δ.*

- *Assume that σ is of the form ς, ς_μ or ω_J. Then G_0/K_0 is a Poisson homogeneous space of group type if and only if the Belavin-Drinfeld triple is trivial and $(\mathfrak{g}_0, \delta)$ is as in Table 1.*
- *Assume that σ is of the form $\omega_{\mu,J}$ with $\mu \neq \mathrm{id}$ and that the Belavin-Drinfeld triple is trivial. Then G_0/K_0 is a Poisson homogeneous space of group type if and only if $\mathfrak{g}_0 = \mathfrak{sl}(3,\mathbb{R})$ and $\lambda_{\alpha,\beta} = -\overline{\lambda_{\mu(\alpha),\mu(\beta)}}$.*

Here $\lambda_{\alpha,\beta} \in \mathbb{C}$ is obtained from $\lambda - \lambda^\dagger = \sum_{\alpha,\beta\in\Delta} \lambda_{\alpha,\beta} h_\alpha \wedge h_\beta$, where λ is the continuous parameter and $\lambda^\dagger$ denotes the transpose of λ. The proof of the Theorem follows from Propositions 2.4 (for $\sigma = \varsigma$, $\Gamma_1 = \Gamma_2 = \emptyset$), 2.5 (for $\sigma = \varsigma_\mu$, $\Gamma_1 = \Gamma_2 = \emptyset$), 2.6 (for $\sigma = \varsigma$ or ς_μ, $\Gamma_1 \neq \emptyset$), 2.7 (for $\sigma = \omega_J$) and 2.12 (for $\sigma = \omega_{\mu,J}$, $\Gamma_1 = \Gamma_2 = \emptyset$), in presence of the information in [AJ, Tables 1.1 and 2.1]– summarized in Proposition 1.8. The only case that remains open is when $\sigma = \omega_{\mu,J}$, $\mu \neq \mathrm{id}$, and the Belavin-Drinfeld triple is non-trivial.

The paper is organized as follows. Section 1 is devoted to preliminaries on Lie bialgebras, including the celebrated theorem of Belavin and Drinfeld, and the classification result in [AJ]. After this, we prove the main result in Section 2, by a case-by-case analysis.

1. Lie bialgebras

$\mathfrak{g}_0$	σ	**Type**	**Continuous parameter**	**Remarks**
$\mathfrak{g}_{\mathbb{R}}$	ς	all	$\lambda_{\alpha,\beta}=0$	
$\mathfrak{su}(n,n+1)$ $\mathfrak{su}(n+1,n+1)$ $\mathfrak{so}(n-1,n+1)$ EII	ς_μ, $\mu\neq \mathrm{id}$	A_{2n} A_{2n+1} D_n E_6	$\lambda_{\alpha,\beta}=\overline{\lambda_{\mu(\alpha),\mu(\beta)}}$, $\mathrm{Re}(\lambda_{\alpha,\beta}+\lambda_{\alpha,\mu(\beta)})=0$	
$\mathfrak{su}(j,n+1-j)$ $\mathfrak{so}(2,2n-1)$ $\mathfrak{sp}(n,\mathbb{R})$ $\mathfrak{so}(2,2n-2)$ $\mathfrak{so}^*(2n)$ $EIII$ $EVII$	ω_J	A_n B_n C_n D_n D_n E_6 E_7	$\lambda_{\alpha,\beta}\in i\mathbb{R}$	**Painted roots:** j^{th} root first root n^{th} root first root n^{th} root extreme of the long branch extreme of the long branch

Explanation of the table. σ the involution defined by $\mathfrak{g}_0$, as in (1.8), (1.9). The painted roots are classifiers of Vogan classification. Explanation of Vogan classification an notations are in [Kn].

TABLE 1. G_0/K_0 Poisson homogeneous space of group type, K_0 adapted

1.1. **Simple Lie algebras.** In this section we introduce the notation that will be used through all the paper. If θ is a bijection of a set X, then X^θ denotes the fixed-point set of θ. If $a\in\mathbb{C}$, we denote by $\overline{a}$ the conjugate of a. We set $i=\sqrt{-1}$. All the Lie algebras in this paper are finite-dimensional, unless explicitly stated.

We denote by $\mathfrak{g}$ a simple complex Lie algebra and by $B(\,,\,):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ the Killing form on $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\lambda\in h^*$, we denote h_λ the element of $\mathfrak{h}$ that satisfies $B(h_\lambda,h)=\lambda(h)$, for all h in $\mathfrak{h}$. We extend $B(\,,\,)$ to $\mathfrak{h}^*\times\mathfrak{h}^*$:

$$B(\lambda,\mu):=B(h_\lambda,h_\mu),\quad \text{for } \lambda,\mu\in h^*.$$

We denote by $\Phi=\Phi(\mathfrak{g},\mathfrak{h})$ the corresponding root system. Let $\Delta\subset\Phi$ be a system of simple roots. Let Φ^+ be the set of positive roots with respect to Δ. Given $\alpha=\sum_{\beta\in\Delta}n_\beta\beta\in\Phi^+$, we denote by $\ell(\alpha)=\sum_{\beta\in\Delta}n_\beta$ the length of α.

Let $\mathfrak{g}_\alpha$ be the root space corresponding to α. Then $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{h}\oplus\mathfrak{g}_-$ is the root space decomposition of $\mathfrak{g}$, where $\mathfrak{g}_\pm=\oplus_{\alpha\in\pm\Phi^+}\mathfrak{g}_\alpha$.

We choose root vectors $e_\alpha \in \mathfrak{g}_\alpha - 0$ such that

$$B(e_\alpha, e_{-\alpha}) = 1, \qquad \text{for } \alpha \in \Phi. \tag{1.1}$$

Then

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \tag{1.2}$$

$$[e_\alpha, e_\beta] = 0, \qquad \text{if } \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \Phi. \tag{1.3}$$

For every $\alpha, \beta \in \Phi$ such that $\alpha + \beta \neq 0$, let $N_{\alpha,\beta}$ defined by

$$[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}, \qquad \text{if } \alpha+\beta \in \Phi. \tag{1.4}$$

We set $N_{\alpha,\beta} = 0$, if $\alpha+\beta \neq 0$ and $\alpha+\beta \notin \Phi$. Thus, we have for all α, β and γ in Φ such that $\alpha+\beta \neq 0$,

$$N_{\alpha,\beta} = -N_{\beta,\alpha}, \tag{1.5}$$

$$N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}, \qquad \text{if } \alpha,\beta,\gamma \in \Phi,\ \alpha+\beta+\gamma = 0. \tag{1.6}$$

The following fact is well-known.

Lemma 1.1. *Let $\alpha, \beta \in \Phi$ such that $\alpha+\beta \in \Phi$, and $\alpha - \beta \notin \Phi$. Then*

$$N_{\alpha,\beta}\, N_{-\alpha,-\beta} = B(\alpha,\beta)^{-1}. \tag{1.7}$$

Proof. If we denote $v = [e_\alpha, e_\beta]$, then

$$\begin{aligned}
[[e_\alpha, e_\beta], [e_{-\alpha}, e_{-\beta}]] &= -[e_{-\alpha}, [e_{-\beta}, v]] - [e_{-\beta}, [v, e_{-\alpha}]], \\
[e_{-\beta}, v] &= -[e_\alpha, [e_\beta, e_{-\beta}]] = -[e_\alpha, h_\beta] = B(\alpha,\beta) e_\alpha \\
[v, e_{-\alpha}] &= -[[e_{-\alpha}, e_\alpha], e_\beta] = -[h_{-\alpha}, e_\beta] = B(\alpha,\beta) e_\beta.
\end{aligned}$$

Set $c = \frac{1}{N_{\alpha,\beta}\, N_{-\alpha,-\beta}}$. Then

$$\begin{aligned}
h_{\alpha+\beta} = [e_{\alpha+\beta}, e_{-\alpha-\beta}] &= \frac{1}{N_{\alpha,\beta}\, N_{-\alpha,-\beta}} [[e_\alpha, e_\beta], [e_{-\alpha}, e_{-\beta}]] \\
&= -c\,([e_{-\alpha}, [e_{-\beta}, v]] + [e_{-\beta}, [v, e_{-\alpha}]]) = -c\, B(\alpha,\beta)([e_{-\alpha}, e_\alpha] + [e_{-\beta}, e_\beta]) = c\, B(\alpha,\beta)\, h_{\alpha+\beta},
\end{aligned}$$

and the result follows. □

In the follows, trough all this work, $\mathfrak{g}$ will denote a simple complex Lie algebra and $B(\ ,\ )$ the Killing form corresponding to $\mathfrak{g}$. Also, $\mathfrak{h}$ denotes a Cartan subalgebra of $\mathfrak{g}$, $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ the corresponding root system, Δ a set of simple roots, Φ^+ the set of positive roots with respect to Δ, $\mathfrak{g}_\alpha$ the root space corresponding to $\alpha \in \Phi$ and $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$ the root space decomposition.

1.2. **Absolutely simple real Lie algebras.** We now describe the real Lie algebras we shall work with. A finite-dimensional real Lie algebra is called *absolutely simple* if its complexification is a simple complex Lie algebra. It is well-known that a simple real Lie algebra is either absolutely simple or is the realification of a complex simple Lie algebra.

Let $\mathfrak{g}$ be a simple complex Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra, Φ be the root system and Δ be a system of simple roots. Let $\mu : \Delta \to \Delta$ be an automorphism of the Dynkin diagram of order 1 or 2. We choose $e_\alpha \in \mathfrak{g}_\alpha - 0$ such that $B(e_\alpha, e_{-\alpha}) = 1$ for $\alpha \in \Phi$. Let J be any subset of the set Δ^μ of simple roots fixed by μ; let $\chi_J : \Delta \to \{0, 1\}$ be the characteristic function of J. Then there exist unique conjugate-linear Lie algebra involutions ς_μ, $\omega_{\mu,J}$ of $\mathfrak{g}$ given respectively by

$$\varsigma_\mu(e_\alpha) = e_{\mu(\alpha)}, \qquad \varsigma_\mu(e_{-\alpha}) = e_{-\mu(\alpha)}, \tag{1.8}$$

$$\omega_{\mu,J}(e_\alpha) = (-1)^{\chi_J(\alpha)} e_{-\mu(\alpha)}, \qquad \omega_{\mu,J}(e_{-\alpha}) = (-1)^{\chi_J(-\alpha)} e_{\mu(\alpha)}, \tag{1.9}$$

for all $\alpha \in \Delta$. Necessarily,

$$\varsigma_\mu(h_\alpha) = h_{\mu(\alpha)}, \qquad \omega_{\mu,J}(h_\alpha) = -h_{\mu(\alpha)}. \tag{1.10}$$

We shall write

$$\varsigma = \varsigma_{\mathrm{id}}, \qquad \omega_J = \omega_{\mathrm{id},J}, \qquad \omega = \omega_{\mathrm{id},\Delta}.$$

Thus ω is the Chevalley involution of $\mathfrak{g}$, with respect to $\mathfrak{h}$ and Δ, and the fixed point set of ω is a compact form $\mathfrak{u}_0$ of $\mathfrak{g}$. Since $\omega(e_\alpha) = -e_{-\alpha}$ for all $\alpha \in \Phi$– see (1.14) below– one has:

$$\mathfrak{u}_0 = \sum_{\alpha\in\Phi} \mathbb{R}(i\, h_\alpha) + \sum_{\alpha\in\Phi} \mathbb{R}(e_\alpha - e_{-\alpha}) + \sum_{\alpha\in\Phi} \mathbb{R} i\, (e_\alpha + e_{-\alpha}). \tag{1.11}$$

The following lemma – a variation of [AJ, Lemma 2.1]– will be useful later.

Lemma 1.2. *Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a conjugate-linear Lie algebra involution such that $\sigma(\mathfrak{h}) = \mathfrak{h}$. Thus, we can define $\sigma^* : \mathfrak{h}^* \to \mathfrak{h}^*$, the adjoint of σ, by $\sigma^*(\lambda)(h) = \lambda(\sigma(h))$, for $\lambda \in H^*$, $h \in \mathfrak{h}$. Then $\sigma^*(\Phi) = \Phi$ and $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_{\sigma^*(\alpha)}$, see* loc. cit. *Then, there exists a choice of non-zero root vectors $e_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$, satisfying* (1.1)*, such that*

(a). If $\sigma^*(\Delta) = \Delta$, there exists a unique automorphism $\mu : \Delta \to \Delta$ of the Dynkin diagram of order 1 or 2 (which does not depend on the choice of the e_α's), such that $\sigma = \varsigma_\mu$ and

$$\varsigma_\mu(e_\alpha) = e_{\mu(\alpha)}, \tag{1.12}$$

$$\varsigma_\mu(h_\alpha) = h_{\mu(\alpha)}, \tag{1.13}$$

for all $\alpha \in \Phi$, where $\mu : \Phi \to \Phi$ is the linear extension of μ.

(b). If $\sigma^*(\Delta) = -\Delta$, the there exists a unique automorphism $\mu : \Delta \to \Delta$ of the Dynkin diagram of order 1 or 2 and a unique subset $\tilde{J}$ of Φ^μ (neither μ nor $\tilde{J}$ depend on the choice of the e_α's) such that: $\sigma = \omega_{\mu,J}$ with $J = \tilde{J} \cap \Delta$. Furthermore,

$$\omega_{\mu,J}(e_\alpha) = (-1)^{\chi_{\tilde{J}}(\alpha)} e_{-\mu(\alpha)}, \tag{1.14}$$

$$\omega_{\mu,J}(h_\alpha) = -h_{\mu(\alpha)}, \tag{1.15}$$

for all $\alpha \in \Phi$, where $\mu : \Phi \to \Phi$ is the linear extension of μ and $\chi_{\tilde{J}} : \Phi \to \{0,1\}$ the characteristic function of $\tilde{J}$.

(c). Assume the situation in (b). Let $\alpha, \beta \in \Phi$ and $N_{\alpha,\beta}$ be defined as in subsection 1.1. We have

(1)

$$\chi_{\tilde{J}}(\alpha) = \chi_{\tilde{J}}(-\alpha) = \chi_{\tilde{J}}(\mu(\alpha)), \tag{1.16}$$

$$(-1)^{\chi_{\tilde{J}}(\alpha)+\chi_{\tilde{J}}(\beta)} N_{-\mu(\alpha),-\mu(\beta)} = (-1)^{\chi_{\tilde{J}}(\alpha+\beta)} \overline{N_{\alpha,\beta}}. \tag{1.17}$$

(2) *Let $\chi_{\mathbb{Z}J} : \Phi \to \mathbb{Z}$ be the linear extension of $\chi_J : \Delta \to \{0,1\}$. Then*

$$(-1)^{\chi_{\tilde{J}}(\alpha)} = (-1)^{\chi_{\mathbb{Z}J}(\alpha)+\ell(\alpha)+1} \tag{1.18}$$

for all $\alpha \in \Phi$.

Proof. Assume that $\sigma^*(\Delta) = \pm\Delta$. Let $\mu : \Delta \to \Delta$ be given by $\mu = \pm\sigma^*$, according to the case. Then μ is an automorphism of the Dynkin diagram, and clearly it has order 1 or 2. Let $f_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$, be any choice of non-zero root vectors satisfying $B(f_\alpha, f_{-\alpha}) = 1$. Let $c_\alpha \in \mathbb{C} - 0$ be such that $\sigma(f_\alpha) = c_\alpha f_{\sigma^*(\alpha)}$, $\alpha \in \Phi$. It is known that $B(x,y) = \overline{B(\sigma(x),\sigma(y))}$ for all x, $y \in \mathfrak{g}$, see [H, p. 180]. Then, for all $\alpha \in \Phi$, we have $1 = B(f_\alpha, f_{-\alpha}) = \overline{B(\sigma(f_\alpha), \sigma(f_{-\alpha}))} = \overline{c_\alpha c_{-\alpha} B(f_{\sigma^*(\alpha)}, f_{-\sigma^*(\alpha)})} = \overline{c_\alpha c_{-\alpha}}$. Hence, if $\alpha \in \Phi$, we have

$$c_\alpha c_{-\alpha} = 1. \tag{1.19}$$

Now, $f_\alpha = \sigma^2(f_\alpha) = \sigma(c_\alpha f_{\sigma^*(\alpha)}) = \overline{c_\alpha}\sigma(f_{\sigma^*(\alpha)}) = \overline{c_\alpha} c_{\sigma^*\alpha} f_\alpha$, thus

$$c_\alpha \overline{c_{\sigma^*\alpha}} = 1. \tag{1.20}$$

We prove (a). In this case $\sigma^* = \mu$. For $\alpha \in \Phi^+$, define d_α, by $c_\alpha = \overline{(d_\alpha)^{-1}} d_{\mu(\alpha)}$. The existence of such d_α is clear. For $\alpha \in \Phi$ let $e_\alpha = d_\alpha f_\alpha$, $e_{-\alpha} = (d_\alpha)^{-1} f_{-\alpha}$. Then $B(e_\alpha, e_{-\alpha}) = 1$ and

$$\sigma(e_\alpha) = \overline{d_\alpha} c_\alpha (d_{\mu(\alpha)})^{-1} e_{\mu(\alpha)} = e_{\mu(\alpha)}, \qquad \sigma(e_{-\alpha}) = \overline{(d_\alpha)^{-1}} c_{-\alpha} (d_{\mu(\alpha)}) e_{-\mu(\alpha)} = e_{-\mu(\alpha)}.$$

The second formula follows from (1.19). The uniqueness of μ is evident and (a) follows.

Let us now prove (b). In this case $\sigma^* = -\mu$. Let $\alpha \in \Phi^+$. If $e_\alpha = d_\alpha f_\alpha$, $e_{-\alpha} = (d_\alpha)^{-1} f_{-\alpha}$, for non-zero scalars d_α, we have

$$\sigma(e_\alpha) = \overline{d_\alpha} c_\alpha d_{\mu(\alpha)} e_{-\mu(\alpha)} \quad \text{and} \quad \sigma(e_{-\alpha}) = \overline{(d_\alpha)^{-1}} c_{-\alpha} (d_{\mu(\alpha)})^{-1} e_{\mu(\alpha)}. \tag{1.21}$$

Assume that $\mu(\alpha) \neq \alpha$ with α positive. Choose d_α, $d_{\mu(\alpha)}$ such that $c_\alpha = (\overline{d_\alpha} d_{\mu(\alpha)})^{-1}$. Then $\sigma(e_\alpha) = e_{-\mu(\alpha)}$, but also $\sigma(e_{-\alpha}) = e_{\mu(\alpha)}$ by (1.19). Hence $\sigma(e_{\pm\mu(\alpha)}) = e_{\mp\alpha}$.

Assume that $\mu(\alpha) = \alpha$. In this case we have $c_\alpha \in \mathbb{R}$ by (1.19) and (1.20). Let $\tilde{J} = \{\beta \in \Phi : c_\beta < 0\}$. For equation (1.21) we get

$$\sigma(e_{\pm\alpha}) = |d_\alpha|^{\pm 2} c_{\pm\alpha} e_{\mp\alpha}$$

Thus, is clear that we can choose $d_\alpha \in \mathbb{R}$ such that $c_\alpha d_\alpha^2 = (-1)^{\chi_{\tilde{J}}(\alpha)}$ and it is easy to see that $\sigma(e_{\pm\alpha}) = (-1)^{\chi_{\tilde{J}}(\pm\alpha)} e_{\mp\alpha}$.

Now we see the uniqueness of $\tilde{J}$: let $f'_\alpha = c f_\alpha$, then $f'_{-\alpha} = (1/c) f_{-\alpha}$ because the pair $f'_\alpha, f'_{-\alpha}$ must satisfy $B(f'_\alpha, f'_{-\alpha}) = 1$. As $\mu(\alpha) = \alpha$, we have that $\sigma(f'_\alpha) = c'_\alpha f'_{-\alpha}$. On the other hand $\sigma(f'_\alpha) = \overline{c}\, \sigma(f_\alpha) = \overline{c}\, c_\alpha\, f_{-\alpha} = \overline{c}\, c\, c_\alpha\, f'_{-\alpha} = |c|^2\, c_\alpha\, f'_{-\alpha}$. Thus, the sign of c_α is equal to to the sign of c'_α, which implies that $\tilde{J}$ is unique, that is, it does not depend on the f_α's. This shows (b).

We prove (c)(1). The equality (1.16) is evident; we obtain (1.17) applying σ to both sides of the equation $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$.

We prove (c)(2). At the beginning of the proof of this lemma, we can choose the non-zero root vectors $f_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$ such that if $[f_\alpha, f_\beta] = M_{\alpha,\beta} f_{\alpha+\beta}$ for all $\alpha, \beta, \alpha+\beta \in \Phi$, then $M_{\alpha,\beta}$ are real and $M_{\alpha,\beta} = -M_{-\alpha,-\beta}$, *cf.* [Kn, Th. 6.6]. Let $\gamma, \beta \in \Phi^+$, such that $\alpha = \gamma + \beta \in \Phi^+$, thus $[e_\gamma, e_\beta] = N_{\gamma,\beta} e_\alpha$ with

$$N_{\gamma,\beta} = \frac{d_\gamma d_\beta}{d_\alpha} M_{\gamma,\beta} \quad \text{and} \quad N_{-\gamma,-\beta} = \frac{(d_\gamma)^{-1}(d_\beta)^{-1}}{(d_\alpha)^{-1}} M_{-\gamma,-\beta},$$

so $\dfrac{N_{-\gamma,-\beta}}{\overline{N_{\gamma,\beta}}} = C \dfrac{M_{-\gamma,-\beta}}{M_{\gamma,\beta}}$ with $C = \dfrac{|d_\alpha|^2}{|d_\gamma|^2 |d_\beta|^2}$, where $|\,|$ denotes the complex norm, so $C > 0$. Thus, we can assume that we have a choice of root vectors as in (b) with $N_{\gamma,\beta}$ such that

$$\frac{N_{-\gamma,-\beta}}{\overline{N_{\gamma,\beta}}} < 0, \tag{1.22}$$

for $\gamma, \beta \in \Phi^+$ with $\gamma + \beta \in \Phi^+$.

We argue then by induction on the length $\ell(\alpha)$ of α. Equation (1.18) is evident if $\alpha \in \Delta$. If $\alpha \in \Phi^+ - \Delta$, fix a decomposition $\alpha = \gamma + \beta$, where $\gamma \in \Phi^+$ and $\beta \in \Delta$. Then

$$\begin{aligned}
(-1)^{\chi_{\tilde{J}}(\alpha)} e_{-\alpha} = \sigma(e_\alpha) &= \frac{1}{\overline{N_{\gamma,\beta}}} [\sigma(e_\gamma), \sigma(e_\beta)] \\
&= \frac{1}{\overline{N_{\gamma,\beta}}} (-1)^{\chi_{\mathbb{Z}J}(\gamma) + \ell(\gamma) + \chi_{\mathbb{Z}J}(\beta) + \ell(\beta)} [e_{-\gamma}, e_{-\beta}] \\
&= \frac{N_{-\gamma,-\beta}}{\overline{N_{\gamma,\beta}}} (-1)^{\chi_{\mathbb{Z}J}(\alpha) + \ell(\alpha)} e_{-\alpha}.
\end{aligned}$$

Then $\dfrac{N_{-\gamma,-\beta}}{\overline{N_{\gamma,\beta}}} = -1$ and (c)(2) follows.

$\square$

From completeness we state this theorem of E. Cartan:

Theorem 1.3. *Let $\mathfrak{g}_0$ be a absolutely simple Lie algebra and $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$. Let σ be the conjugate-linear Lie algebra involution such that $\mathfrak{g}_0 = \mathfrak{g}^\sigma$. Then, there exists $\mathfrak{h}$ a σ-invariant Cartan subalgebra of $\mathfrak{g}$, a system of simple roots Δ with $\sigma^*(\Delta) = \pm\Delta$ and an election of $e_\alpha \in \mathfrak{g}_\alpha - \{0\}$ ($\alpha \in \Phi$); such that σ is an involution of the form $\varsigma, \varsigma_\mu, \omega_J$ or $\omega_{\mu,J}$ with $\mu \neq \mathrm{id}$ and $J \subset \Delta^\mu$, but $\sigma \neq \omega$ (if non-compact), or $\mathfrak{g}_0 = \mathfrak{g}^\omega = \mathfrak{u}_0$ (if compact).*

Let's denote σ an involution of the form $\varsigma, \varsigma_\mu, \omega_J$ or $\omega_{\mu,J}$ with $\mu \neq \mathrm{id}$ and $J \subset \Delta^\mu$, but $\sigma \neq \omega$. We denote $\mathfrak{g}_0 = \mathfrak{g}^\sigma$ and $\mathfrak{h}_0 = \mathfrak{h}^\sigma$ the set of fixed points by σ on $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

An easy calculation shows that σ commutes with ω, so ω preserves $\mathfrak{g}_0$. Let $\theta_0 : \mathfrak{g}_0 \to \mathfrak{g}_0$ be the linear Lie algebra involution given by the restriction of ω. Then θ_0 is a Cartan involution of $\mathfrak{g}_0$ and $\theta_0(\mathfrak{h}_0) = \mathfrak{h}_0$. Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be the complexification of θ_0; clearly $\theta = \sigma\omega$. The transpose of θ respect to the Killing form preserves Δ, and in fact coincides with μ. Let

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0, \quad \text{resp.} \quad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \tag{1.23}$$

be the Cartan decomposition associated to θ_0, resp. θ. It is known that $\mathfrak{k}_0 = \mathfrak{u}_0 \cap \mathfrak{g}_0$. Let G_0 (resp. K_0) be the connected, simple connected Lie group (resp. connected Lie subgroup of G_0) with Lie algebra $\mathfrak{g}_0$ (resp. $\mathfrak{k}_0$).

1.3. **Description of $\mathfrak{k}$.** To prove the main result we will need an explicit expression of $\mathfrak{k}$. In order to do this, we first write down an explicit form of σ and θ.

In the hypothesis of the lemma 1.2, if $\sigma = \omega_{\mu,J}$, we denote $\tilde{J}$, the unique subset of Φ^μ determined by lemma 1.2(b). In the follows, we say that *$\tilde{J}$ is determined by $\omega_{\mu,J}$*.

Let $\gamma \in \Phi$. As $\theta = \omega\sigma$ and $\omega(e_\alpha) = -e_{-\alpha}$, we have, *cf.* (1.10):

(1.24) If $\sigma = \varsigma_\mu$ then $\theta(e_\gamma) = -e_{-\mu(\gamma)}, \quad \theta(h_\gamma) = -h_{\mu(\gamma)}$.

(1.25) If $\sigma = \omega_{\mu,J}$ then $\theta(e_\gamma) = -(-1)^{\chi_{\tilde{J}}(\gamma)} e_{\mu(\gamma)}, \quad \theta(h_\gamma) = h_{\mu(\gamma)}$.

Since $\mathfrak{k} = \mathfrak{g}^\theta$, an easy calculations shows:

Lemma 1.4. *Let σ be an involution of the form $\varsigma, \varsigma_\mu, \omega_J$, or $\omega_{\mu,J}$ with $\mu \neq \mathrm{id}$ and $J \subset \Delta^\mu$. Let $\mathfrak{k}$ and $\mathfrak{k}_0$ be as in* (1.23)*. Then*

(1.26)
If $\sigma = \varsigma$ *then* $\mathfrak{k} = \langle e_\alpha - e_{-\alpha} : \alpha \in \Phi \rangle$.

(1.27)
If $\sigma = \varsigma_\mu$ *then* $\mathfrak{k} = \langle h_\alpha - h_{\mu(\alpha)},\ e_\alpha - e_{-\mu(\alpha)} : \alpha \in \Phi \rangle$.

(1.28)
If $\sigma = \omega_J$, *let* $\tilde{J}$ *the subset of* Φ^μ *determined by* ω_J. *Then* $\mathfrak{k} = \langle h_\alpha,\ \big(1 - (-1)^{\chi_{\tilde{J}}(\alpha)}\big) e_\alpha : \alpha \in \Phi \rangle$.

(1.29)
If $\sigma = \omega_{\mu,J}$, *let* $\tilde{J}$ *the subset of* Φ^μ *determined by* $\omega_{\mu,J}$. *Then* $\mathfrak{k} = \langle h_\alpha + h_{\mu(\alpha)},\ e_\alpha - (-1)^{\chi_{\tilde{J}}(\alpha)} e_{\mu(\alpha)} : \alpha \in \Phi \rangle$.

□

1.4. **Lie bialgebras.** As is well-known, there is a one-to-one correspondence between Poisson-Lie group structures on a connected and simply connected Lie group A and Lie bialgebra structures on its Lie algebra $\mathfrak{a}$ [D1]. Recall that a (real or complex) Lie bialgebra is a pair $(\mathfrak{a}, \delta)$ where $\mathfrak{a}$ is a (real or complex) Lie algebra and $\delta : \mathfrak{a} \to \Lambda^2(\mathfrak{a})$ is a 1-cocycle satisfying the co-Jacobi identity.

The Drinfeld double of a Lie bialgebra $\mathfrak{a}$ is denoted $D(\mathfrak{a})$– see [D2]. Let $(\mathfrak{a}, \delta)$ be a quasitriangular Lie bialgebra, *i. e.* there exists $r = \sum_i r_i \otimes r^i \in \mathfrak{a} \otimes \mathfrak{a}$, such that $\delta = \partial r$ and r satisfies the classical Yang-Baxter equation (CYBE):

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

where if $r^{12} = \sum_i r_i \otimes r^i \otimes 1$, $r^{13} = \sum_i r_i \otimes 1 \otimes r^i$, etc. In this case, we denote $(\mathfrak{a}, r)$ instead of $(\mathfrak{a}, \delta)$. We denote $r^\dagger$ de transposition of r, i.e. $r^\dagger = \sum_i r^i \otimes r_i$. Let $r_s = r + r^\dagger \in S^2\mathfrak{a}$, then we can define a bilinear form $\langle\ ,\ \rangle_*$ on $\mathfrak{a}^*$ and a map $r_s : \mathfrak{a}^* \to \mathfrak{a}$ by

$$\langle \mu, \gamma \rangle_* = \sum_i \mu(s_i)\gamma(s^i), \qquad r_s(\mu) = \sum_i \mu(s_i) s^i,$$

where $r_s = \sum_i s_i \otimes s^i$, $\mu, \gamma \in \mathfrak{a}^*$. We say that $(\mathfrak{a}, r)$ is *factorizable* if $\langle\ ,\ \rangle_*$ is a *nondegenerate* inner product on $\mathfrak{a}^*$ [RS]. In this case, the map r_s is an isomorphism and the bilinear from on $\mathfrak{a}$ defined

$$\langle x, y \rangle = \langle r_s(x)^{-1}, r_s(y)^{-1}\gamma \rangle_*, \qquad x, y \in \mathfrak{a}$$

is nondegenerate. An easy calculation shows that

Proposition 1.5. *(*[RS]*) If* $(\mathfrak{a}, r)$ *is a factorizable Lie bialgebra, then* $D(\mathfrak{a})$ *is isomorphic to* $\mathfrak{a} \oplus \mathfrak{a}$, *with bilinear form given by*

$$\langle (x, x'), (y, y') \rangle = \langle x, y \rangle - \langle x', y' \rangle.$$

Turning to the real case, a real Lie bialgebra $(\mathfrak{a}_0, \delta)$ is *almost factorizable* if the complexification $(\mathfrak{a}, \delta)$ is factorizable ([AJ]). The following class of almost factorizable Lie bialgebras is important in the classification of real simple Lie bialgebras given in [AJ].

Definition 1.6. We say that $(\mathfrak{a}_0, \delta)$ is *imaginary factorizable* if the complexification $(\mathfrak{a}, \delta)$ is factorizable and $r \in \mathfrak{a} \otimes \mathfrak{a}$ is given by

$$r = r_\Lambda + i r_\Omega, \quad \text{where } r_\Lambda \in \Lambda^2(\mathfrak{a}_0), \quad r_\Omega \in S^2(\mathfrak{a}_0). \tag{1.30}$$

In this case, $D(\mathfrak{a}_0)$ is isomorphic to the realification $\mathfrak{a}^{\mathbb{R}}$, with bilinear form given by $(u|v) = 2\,\mathrm{Re}\langle u, v\rangle = \langle u, v\rangle - \langle \sigma(u), \sigma(v)\rangle$ [AJ, Prop. 3.1]. Here σ is the conjugate-linear involution of $\mathfrak{a}$ such that $\mathfrak{a}_0 = \mathfrak{a}^\sigma$.

We review now the celebrated classification of Lie bialgebra structures on complex simple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and Δ a system of simple roots. Recall that a *Belavin-Drinfeld triple* (BD-triple for short) is a triple (Γ_1, Γ_2, T) where Γ_1, Γ_2 are subsets of Δ and $T : \Gamma_1 \to \Gamma_2$ is a bijection that preserves the inner product coming from the Killing form, such that the *nilpotency condition* holds: for any $\alpha \in \Gamma_1$ there exists a positive integer n for which $T^n(\alpha)$ belongs to Γ_2 but not to Γ_1.

Let (Γ_1, Γ_2, T) be a Belavin–Drinfeld triple. We can extend $T : \mathbb{Z}\Gamma_1 \to \mathbb{Z}\Gamma_2$ in the natural way. Let $L \subset \Delta$, define $\mathfrak{g}_L = \bigoplus_{\alpha\in\mathbb{Z}L} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha\in L} \mathbb{C}h_\alpha$. Then we can define $\hat{T} : \mathfrak{g}_{\Gamma_1} \to \mathfrak{g}_{\Gamma_2}$ a Lie algebra automorphism, such that $\hat{T}(\mathfrak{g}_\alpha) = \mathfrak{g}_{T(\alpha)}$ for all $\alpha \in \mathbb{Z}\Gamma_1$– see [BD].

Let $\widehat{\Gamma}_i = \mathbb{Z}\Gamma_i \cap \Phi^+$ for $i = 1, 2$. There is an associated partial ordering on Φ^+ given by $\alpha \prec \beta$ if $\alpha \in \widehat{\Gamma}_1$, $\beta \in \widehat{\Gamma}_2$, and $\beta = T^n(\alpha)$ for an integer $n > 0$.

We denote by $\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ the Casimir element of $\mathfrak{g}$; that is, $\Omega = \sum x_i \otimes x^i$ where (x_i), (x^i) is any pair of dual basis with respect to the Killing form of $\mathfrak{g}$. We denote by Ω_0 the component in $\mathfrak{h} \otimes \mathfrak{h}$ of Ω; that is, $\Omega_0 = \sum h_i \otimes h^i$ where (h_i), (h^i) is any pair of dual basis in $\mathfrak{h}$ with respect to the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{h}$.

A *continuous parameter* for the BD-triple (Γ_1, Γ_2, T) is an element $\lambda \in \mathfrak{h}^{\otimes 2}$ such that

$$(T(\alpha) \otimes 1)\lambda + (1 \otimes \alpha)\lambda = 0, \quad \text{for all } \alpha \in \Gamma_1, \tag{1.31}$$

$$\lambda + \lambda^\dagger = \Omega_0. \tag{1.32}$$

Recall that $\lambda^\dagger$ denotes the transposition of λ.

Theorem 1.7 (Belavin–Drinfeld, see [BD])**.** *Let $(\mathfrak{g}, \delta)$ be a factorizable complex simple Lie bialgebra. Then there exist a Cartan subalgebra $\mathfrak{h}$, a system of simple roots Δ, a Belavin–Drinfeld triple (Γ_1, Γ_2, T), a continuous parameter λ and $t \in \mathbb{C} - 0$ such that the r-matrix is given by*

$$r = \frac{t}{2}\left(\lambda + \sum_{\alpha\in\Phi^+} x_{-\alpha} \otimes x_\alpha + \sum_{\alpha,\beta\in\Phi^+,\alpha\prec\beta} x_{-\alpha} \wedge x_\beta\right), \tag{1.33}$$

where $x_\alpha \in \mathfrak{g}_\alpha$ are normalized by

$$B(x_\alpha, x_{-\alpha}) = 1, \qquad \textit{for all } \alpha \in \Phi^+ \tag{1.34}$$

$$\hat{T}(x_\alpha) = x_{T(\alpha)}, \qquad \textit{for all } \alpha \in \Gamma_1. \tag{1.35}$$

Clearly, $r + r^\dagger = t\Omega$. Note that the normalization condition (1.34) is the same as (1.1). Thus, given any family $\{e_\alpha : \alpha \in \Phi\}$ satisfying (1.1), there exists $C_\alpha \in \mathbb{C}$ such that

$$x_\alpha = C_\alpha e_\alpha, \qquad C_\alpha C_{-\alpha} = 1.$$

We next recall some results of [AJ] about the classification of real simple Lie bialgebras. Let μ be an automorphism of the Dynkin diagram.

- A BD-triple (Γ_1, Γ_2, T) is *μ-stable* if $\mu(\Gamma_1) = \Gamma_1$, $\mu(\Gamma_2) = \Gamma_2$, and $T\mu = \mu T$.
- A BD-triple (Γ_1, Γ_2, T) is *μ-antistable* if $\mu(\Gamma_1) = \Gamma_2$, $\mu(\Gamma_2) = \Gamma_1$, and $T^{-1}\mu = \mu T$.

If $\mu = \mathrm{id}$ then all BD-triples are μ-stable, and the only BD-triple μ-antistable has $\Gamma_1 = \Gamma_2 = \emptyset$.

Proposition 1.8. *Let $(\mathfrak{g}_0, \delta)$ be an absolutely simple real Lie bialgebra. Let $\mathfrak{g}$ be the complexification of $\mathfrak{g}_0$ and let σ be the conjugate-linear involution of $\mathfrak{g}$ whose fixed-point set is $\mathfrak{g}_0$. Assume that $(\mathfrak{g}_0, \delta)$ is almost factorizable. Then there exist:*

- *A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.*
- *A system of simple roots $\Delta \subset \Phi(\mathfrak{g}, \mathfrak{h})$.*
- *A Belavin-Drinfeld triple (Γ_1, Γ_2, T) and a continuous parameter $\lambda \in \mathfrak{h}^{\otimes 2}$. Write*

$$\lambda - \lambda^\dagger = \sum_{\alpha,\beta \in \Delta} \lambda_{\alpha,\beta} h_\alpha \wedge h_\beta.$$

  By convention $\lambda_{\alpha,\beta} = -\lambda_{\beta,\alpha}$ for all $\alpha, \beta \in \Delta$.

- *A complex number c with $c^2 \in \mathbb{R}$; set $t = 2ic$.*

All these data verify:

(a) $\mathfrak{h}$ is stable under σ (we denote $\mathfrak{h}_0 := \mathfrak{h} \cap \mathfrak{g}_0$).

(b) $\sigma^*(\Delta)$ is either Δ or $-\Delta$; furthermore $\mu := \sigma^* : \Delta \to \pm\Delta$ is an automorphism of the Dynkin diagram. There are two possibilities:

(i) If $\sigma^*(\Delta) = \Delta$ then, by Lemma 1.2, there is an appropriate choice of the $e_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Phi$) satisfying (1.1)*, such that σ is either ς or ς_μ with $\mu \neq \mathrm{id}$ and* (1.12) *holds.*

If $\sigma = \varsigma$, $t \in \mathbb{R}$, then $\lambda_{\alpha,\beta} \in \mathbb{R}$ for all $\alpha, \beta \in \Delta$ (no restrictions on the BD-triple).

If $\sigma = \varsigma_\mu$, then $t \in \mathbb{R}$, $\lambda_{\alpha,\beta} = \overline{\lambda_{\mu(\alpha),\mu(\beta)}}$, for all $\alpha, \beta \in \Delta$ and the BD-triple is μ-stable.

(ii) If $\sigma^*(\Delta) = -\Delta$ then, by Lemma 1.2, there is an appropriate choice of the $e_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Phi$) satisfying (1.1)*, such that σ is either ω_J, ω or $\omega_{\mu,J}$ with $\mu \neq \mathrm{id}$ and $J \subset \Delta^\mu$, and* (1.14) *holds.*

If $\sigma = \omega_J$, or $\sigma = \omega$, then $t \in i\mathbb{R}$, $\lambda_{\alpha,\beta} \in i\mathbb{R}$, for all $\alpha, \beta \in \Delta$ and the BD-triple has $\Gamma_1 = \Gamma_2 = \emptyset$.

If $\sigma = \omega_{\mu,J}$, then $t \in i\mathbb{R}$, $\lambda_{\alpha,\beta} = -\overline{\lambda_{\mu(\alpha),\mu(\beta)}}$, for all $\alpha, \beta \in \Delta$ and the BD-triple is μ-antistable.

(c) $\delta = \partial r$ as in Theorem 1.7. Furthermore $\delta = \partial r_0$ where $r_0 \in \Lambda^2(\mathfrak{g}_0)$ is given by the formula

$$r_0 = \frac{t}{2}\left(\lambda - \lambda^\dagger + \sum_{\alpha \in \Phi^+} e_{-\alpha} \wedge e_\alpha + \sum_{\alpha,\beta \in \Phi^+, \alpha \prec \beta} C_{-\alpha} C_\beta \, e_{-\alpha} \wedge e_\beta\right), \tag{1.36}$$

with an adequate election of $C_\alpha \in \mathbb{C}$ for $\alpha \in \Phi$ such that $C_\alpha C_{-\alpha} = 1$.

(d)

$$(\theta \otimes \theta) r_0 = \frac{\overline{t}}{2}\left(\sum_{\alpha,\beta \in \Delta} \overline{\lambda_{\alpha,\beta}} h_\alpha \wedge h_\beta - \sum_{\alpha \in \Phi^+} e_{-\alpha} \wedge e_\alpha + \sum_{\alpha,\beta \in \Phi^+, \alpha \prec \beta} \overline{C_{-\alpha} C_\beta} \, e_\alpha \wedge e_{-\beta}\right). \tag{1.37}$$

Proof. Parts (a) to (c) are [AJ, Lemma 3.1 and Lemma 3.4] combined with Lemma 1.2. As r_0 belongs to $\Lambda^2(\mathfrak{g}_0)$, $(\sigma \otimes \sigma) r_0 = r_0$. Since $\theta = \omega\sigma$, part (d) follows. □

1.5. **Poisson homogeneous spaces of group type arising from graphs.**

Let A be a connected and simply connected Poisson-Lie group with Lie bialgebra $(\mathfrak{a}, \delta)$. Let T be a connected Lie subgroup of A with Lie algebra $\mathfrak{t}$. Recall that Poisson homogeneous structures on A/T are classified by Lagrangian subalgebras $\mathfrak{l}$ of $D(\mathfrak{a}) = \mathfrak{a} \oplus \mathfrak{a}^*$, the Drinfeld double of $\mathfrak{a}$, such that $\mathfrak{l} \cap \mathfrak{a} = \mathfrak{t}$ [D3].

Recall that the canonical bilinear form of $D(\mathfrak{a})$ is given by $\langle x + \mu | x' + \mu' \rangle = \mu'(x) + \mu(x')$ for $x, x' \in \mathfrak{a}$, $\mu, \mu' \in \mathfrak{a}^*$. In this subsection, if $\mathfrak{v}$ is a subspace of $\mathfrak{a}$, then $\mathfrak{v}^\perp$ denotes the orthogonal subspace with respect to $\langle \, | \, \rangle$, thus $\mathfrak{v}^\perp \cap \mathfrak{a}^* = \{\mu \in \mathfrak{a}^* : \mu(\mathfrak{v}) = 0\}$ is the annihilator of $\mathfrak{v}$.

Lemma 1.9. *If $\mathfrak{v}$ is a subspace of $\mathfrak{a}$, then $\mathfrak{u} = \mathfrak{v} \oplus (\mathfrak{v}^\perp \cap \mathfrak{a}^*)$ is a Lagrangian subspace of $D(\mathfrak{a})$.*

Proof. Since $\mathfrak{v} \subset \mathfrak{a}$, $\mathfrak{v}^\perp \cap \mathfrak{a}^* \subset \mathfrak{a}^*$, and $\mathfrak{a}$, $\mathfrak{a}^*$ are isotropic, we have that $\mathfrak{v}$ and $\mathfrak{v}^\perp \cap \mathfrak{a}^*$ are isotropic. As $\langle \mathfrak{v} | \mathfrak{v}^\perp \cap \mathfrak{a}^* \rangle \subset \langle \mathfrak{v} | \mathfrak{v}^\perp \rangle = 0$, $\mathfrak{u}$ is isotropic. It remains to show that $\mathfrak{u}$ is Lagrangian, or equivalently that $\dim(\mathfrak{u}) = \dim(\mathfrak{a}) =: n$. Because of the non degeneracy of the bilinear form on $D(\mathfrak{a})$ we have that $\dim(\mathfrak{v}) + \dim(\mathfrak{v}^\perp) = \dim(D(\mathfrak{a})) = 2n$. But $\mathfrak{v}^\perp = \mathfrak{a} \oplus (\mathfrak{v}^\perp \cap \mathfrak{a}^*)$, thus $\dim(\mathfrak{v}^\perp) = n + \dim(\mathfrak{v}^\perp \cap \mathfrak{a}^*)$. Hence $\dim(\mathfrak{v}) + \dim(\mathfrak{v}^\perp \cap \mathfrak{a}^*) = n$. □

The following result should be well-known; we give a proof for the sake of completeness.

Proposition 1.10. *A/T is a Poisson homogeneous space of group type if and only if there exists a Lagrangian subalgebra $\mathfrak{u}$ of $D(\mathfrak{a})$ such that*

$$\mathfrak{u} = \mathfrak{t} \oplus (\mathfrak{u} \cap \mathfrak{a}^*). \tag{1.38}$$

Note that (1.38) implies $\mathfrak{u} \cap \mathfrak{a} = \mathfrak{t}$.

Proof. ($\Rightarrow$) Let $\mathfrak{u} = \mathfrak{t} \oplus (\mathfrak{t}^\perp \cap \mathfrak{a}^*) \subset \mathfrak{t}^\perp$. It is clear that (1.38) holds, and $\mathfrak{u}$ is a Lagrangian subspace by the previous Lemma. It remains to verify that $\mathfrak{u}$ is a Lie subalgebra. Now, by hypothesis $\mathfrak{t}$ and $\mathfrak{t}^\perp \cap \mathfrak{a}^* = \{\mu \in \mathfrak{a}^* : \mu(\mathfrak{t}) = 0\}$ are Lie subalgebras of $D(\mathfrak{a})$ (see the Introduction). Let $x \in \mathfrak{t}$, $y \in \mathfrak{t}^\perp \cap \mathfrak{a}^*$ and $z = [x, y]$. If $w \in \mathfrak{t}$, $\langle w|z\rangle = \langle [w, x]|y\rangle \in \langle \mathfrak{t}|\mathfrak{t}^\perp\rangle = 0$. If $w \in \mathfrak{t}^\perp \cap \mathfrak{a}^*$, $\langle w|z\rangle = -\langle [w, y]|x\rangle \in \langle \mathfrak{t}^\perp \cap \mathfrak{a}^*|\mathfrak{t}\rangle = 0$. Thus $\langle z|\mathfrak{u}\rangle = 0$. Since $\mathfrak{u}$ is Lagrangian, we conclude that $z \in \mathfrak{u}$.

($\Leftarrow$) If $x \in \mathfrak{t}^\perp \cap \mathfrak{a}^*$, then $\langle x|\mathfrak{t}\rangle = \langle x|\mathfrak{a}^*\rangle = 0$, hence $\langle x|\mathfrak{u}\rangle = 0$. Thus $\langle \mathfrak{t}^\perp \cap \mathfrak{a}^*|\mathfrak{u}\rangle = 0$, and $\mathfrak{t}^\perp \cap \mathfrak{a}^* \subset \mathfrak{u}$ since $\mathfrak{u}$ is Lagrangian. Hence $\mathfrak{t} \oplus (\mathfrak{t}^\perp \cap \mathfrak{a}^*) \subset \mathfrak{u}$. By Lemma 1.9, $\mathfrak{t} \oplus (\mathfrak{t}^\perp \cap \mathfrak{a}^*)$ is also a Lagrangian subspace. Then $\mathfrak{t} \oplus (\mathfrak{t}^\perp \cap \mathfrak{a}^*) = \mathfrak{u}$, and this implies that $\mathfrak{t}^\perp \cap \mathfrak{a}^* = \mathfrak{u} \cap \mathfrak{a}^*$ is a Lie subalgebra of $\mathfrak{a}^*$. ☐

The following construction of Poisson homogeneous spaces was observed by C. De Concini, and independiently by Karolinsky [K]. Let A be a connected (real or complex) Poisson-Lie group with factorizable Lie bialgebra $\mathfrak{a}$: recall that the Drinfeld double is isomorphic to the Lie algebra $\mathfrak{a} \oplus \mathfrak{a}$, and the invariant form is given by $\langle (x, x'), (y, y')\rangle = \langle x, y\rangle - \langle x', y'\rangle$ for $(x, x'), (y, y') \in \mathfrak{a} \oplus \mathfrak{a}$ (Proposition 1.5). Let $\rho \in \mathrm{Aut}(\mathfrak{a})$ preserving $\langle\, ,\, \rangle$. Then, the graph of ρ, namely $\mathfrak{u}_\rho = \{(x, \rho(x)) : x \in \mathfrak{a}\}$, is a Lagrangian subalgebra of the Drinfeld double and A/T is a Poisson homogeneous space, where T is the connected component of the identity of A^ρ.

This construction can be extended to the imaginary-factorizable case. Let A_0 be a connected real Poisson-Lie group with imaginary factorizable Lie bialgebra $(\mathfrak{a}_0, \delta)$. Let $\mathfrak{a}$ be the complexification of $\mathfrak{a}_0$ and σ the conjugate-linear automorphism of $\mathfrak{a}$ whose fixed point set is $\mathfrak{a}_0$. Let $\theta_0 \in \mathrm{Aut}(\mathfrak{a}_0)$ such that $\theta := \theta_0 \otimes \mathrm{id}$ preserves $\langle\, ,\, \rangle$. Let $\mu = \theta\sigma = \sigma\theta$, a conjugate-linear automorphism of $\mathfrak{a}$. Let H be the connected component of the identity of $A_0^{\theta_0}$.

Proposition 1.11. *Assume that θ_0 is an involution. Then A_0/H is a Poisson homogeneous space.*

Proof. As $\mathfrak{a}_0$ is imaginary factorizable, recall that $D(\mathfrak{a}_0)$ is isomorphic to the realification $\mathfrak{a}^\mathbb{R}$, with bilinear form given by $(u|v) = 2\,\mathrm{Re}\langle u, v\rangle = \langle u, v\rangle - \langle \sigma(u), \sigma(v)\rangle$ (see definition 1.6 and what follows). We will show that the real Lie subalgebra $\mathfrak{m} := (\mathfrak{a}^\mathbb{R})^\mu$ is Lagrangian, so, from the Drinfeld's criterion, A_0/H results a Poisson homogeneous space. If $u, v \in \mathfrak{m}$, then $(u|v) = \langle u, v\rangle - \langle \sigma(u), \sigma(v)\rangle = \langle u, v\rangle - \langle \theta(u), \theta(v)\rangle = 0$, thus $\mathfrak{m}$ is isotropic. Since $\theta\sigma = \sigma\theta$, we have $(\mathfrak{a}^\mathbb{R})^\mu \cap \mathfrak{a}_0 = (\mathfrak{a}^\mathbb{R})^{\sigma\theta} \cap \mathfrak{a}_0 = \mathfrak{a}_0^{\theta_0}$. Also, $\mathfrak{m} = \mathfrak{a}_0^{\theta_0} \oplus i\mathfrak{p}_0$, where $\mathfrak{p}_0$ is the eigenspace of θ_0 of eigenvalue -1. Thus, $\dim \mathfrak{m} = \dim \mathfrak{a}_0^{\theta_0} + \dim \mathfrak{p}_0 = \dim \mathfrak{a}_0$, since θ_0 is an involution, and $\mathfrak{m}$ is Lagrangian. ☐

Note that this Poisson homogeneous space is of group type if and only if $\mathfrak{m} = \mathfrak{a}_0^{\theta_0} \oplus (\mathfrak{m} \cap r_s(\mathfrak{a}_0^*))$, because of Proposition 1.10. For the definition de r_s, see the beginning of subsection 1.4 .

In conclusion, the symmetric spaces G_0/K_0 always bear a structure of Poisson homogeneous space, by Propositions 1.8, 1.10– together with De Concini's remark– and 1.11. In this paper we shall investigate when G_0/K_0 bears a structure of Poisson homogeneous space *of group type.*

2. Proof of the main result

In this section we fix $(\mathfrak{g}_0, \delta)$ an almost factorizable absolutely simple real Lie bialgebra. Let G_0 be a non-compact absolutely simple real Lie group with finite center and Lie algebra $\mathfrak{g}_0$ and K_0 de maximal compact subgroup of G_0 adapted to δ (see the introduction). As usual, $\mathfrak{k}_0$ denotes the Lie algebra of K_0. Let σ be the conjugate-linear involution of $\mathfrak{g}$, the complexification of $\mathfrak{g}_0$, such that $\mathfrak{g}_0 = \mathfrak{g}^\sigma$. Now and at the end of this section we will use the notation of subsection 1.2 and the notation and results of proposition 1.8. Also, we use the description of $\mathfrak{k}$, the complexification of $\mathfrak{k}_0$, given in subsection 1.3.

As we said in the Introduction, G_0/K_0 is a a Poisson homogeneous space of group type of G_0 if and only if $\mathfrak{k}_0$ is a coideal of $\mathfrak{g}_0$. Our goal is to determine when $\mathfrak{k}_0$ is a coideal of $\mathfrak{g}_0$.

Lemma 2.1. *G_0/K_0 is a Poisson homogeneous space of group type if and only if*

$$\operatorname{ad}\mathfrak{k}_0\big((\operatorname{id}-\theta)\otimes(\operatorname{id}-\theta)(r_0)\big) = \operatorname{ad}\mathfrak{k}\big((\operatorname{id}-\theta)\otimes(\operatorname{id}-\theta)(r_0)\big) = 0.$$

Proof. Let $r_0 = r_1 + r_2$ with $r_1 \in \mathfrak{g}_0 \otimes \mathfrak{k}_0 + \mathfrak{k}_0 \otimes \mathfrak{g}_0$ and $r_2 \in \mathfrak{p}_0 \otimes \mathfrak{p}_0$. If $u \in \mathfrak{k}_0$, then $\delta(u) = \operatorname{ad} u\, r_0 = \operatorname{ad} u\, r_1 + \operatorname{ad}\, u r_2$ and $\operatorname{ad} u\, r_1 \in\in \mathfrak{g}_0 \otimes \mathfrak{k}_0 + \mathfrak{k}_0 \otimes \mathfrak{g}_0$ and $\operatorname{ad} u\, r_2 \in \mathfrak{p}_0 \otimes \mathfrak{p}_0$. Hence $\mathfrak{k}_0$ is a coideal if and only if $\operatorname{ad} u\, r_2 = 0$ for all $u \in \mathfrak{k}_0$. Now, if $r_1 = \sum x_i \otimes x^i$ with x_i or x^i in $\mathfrak{k}_0$ and $r_2 = \sum y_i \otimes y^i$ with y_i and y^i in $\mathfrak{p}_0$, because $\mathfrak{k}_0$ acts as id on $\mathfrak{k}_0$ and as $-\operatorname{id}$ on $\mathfrak{p}_0$, we have $(\operatorname{id}-\theta)\otimes(\operatorname{id}-\theta)x_i \otimes x^i = 0$ for all i, and $(\operatorname{id}-\theta)\otimes(\operatorname{id}-\theta)y_i \otimes y^i = 4y_i \otimes y^i$. Thus $(\operatorname{id}-\theta)\otimes(\operatorname{id}-\theta)r_0 = 4r_2$ and $\operatorname{ad} u\,(\operatorname{id}-\theta)\otimes(\operatorname{id}-\theta)r_0 = 4\operatorname{ad} u\, r_2$. Hence $\mathfrak{k}_0$ is a coideal if and only if $\operatorname{ad} u\, r_2 = 0$ for all $u \in \mathfrak{k}_0$ if and only if $\operatorname{ad} u\,(\operatorname{id}-\theta)\otimes(\operatorname{id}-\theta)r_0 = 0$ for all $u \in \mathfrak{k}_0$. □

We next give an explicit expression of

$$\widetilde{r_0} := (\operatorname{id}-\theta)\otimes(\operatorname{id}-\theta)(r_0) = r_0 + (\theta\otimes\theta)(r_0) - (\operatorname{id}\otimes\theta + \theta\otimes\operatorname{id})(r_0)$$

according to the different possibilities for σ. Then we analyze when $\mathfrak{k}_0$ is a coideal case by case.

2.1. Computation of $(\operatorname{id}-\theta)\otimes(\operatorname{id}-\theta)(r_0)$.

In the calculations below, keep in mind the equality $(f\otimes\operatorname{id} + \operatorname{id}\otimes f)(a\wedge b) = f(a)\wedge b + a\wedge f(b)$, $a, b \in V$, $f \in \operatorname{End} V$. Set

$$t_{\alpha,\beta} = 2Re(\lambda_{\alpha,\beta} + \lambda_{\alpha,\mu(\beta)}), \qquad \alpha,\beta\in\Delta, \tag{2.1}$$

$$s_{\alpha,\beta} = 2\,i\,Im(\lambda_{\alpha,\beta} - \lambda_{\alpha,\mu(\beta)}), \qquad \alpha,\beta\in\Delta, \tag{2.2}$$

$$d_{\alpha,\beta} = C_{-\alpha}C_\beta, \qquad \alpha,\beta\in\Phi^+, \quad \alpha\prec\beta. \tag{2.3}$$

Proposition 2.2. *If $\sigma = \varsigma$, then*

$$\widetilde{r_0} = \frac{t}{2}\left(\sum_{\alpha,\beta\in\Delta} 2\lambda_{\alpha,\beta}\, h_\alpha \wedge h_\beta + \sum_{\alpha\prec\beta}(\overline{d_{\alpha,\beta}}\, e_{-\alpha}\wedge e_\beta + d_{\alpha,\beta}\left(e_\alpha \wedge e_{-\beta} + e_\alpha\wedge e_\beta + e_{-\alpha}\wedge e_{-\beta}\right))\right). \tag{2.4}$$

If $\sigma = \varsigma_\mu$, $\mu \neq \mathrm{id}$, then

$$\widetilde{r_0} = \frac{t}{2}\left(\sum_{\alpha,\beta\in\Delta} t_{\alpha,\beta}\, h_\alpha \wedge h_\beta + \sum_{\alpha\prec\beta}\overline{d_{\alpha,\beta}}\, e_\alpha\wedge e_{-\beta} + d_{\alpha,\beta}\left(e_{-\alpha} \wedge e_\beta + e_{\mu(\alpha)}\wedge e_\beta + e_{-\alpha}\wedge e_{-\mu(\beta)}\right))\right). \tag{2.5}$$

If $\sigma = \omega_J$, then

$$\widetilde{r_0} = t\sum_{\alpha\in\Phi^+}(1+(-1)^{\chi_{\bar{J}}(\alpha)})e_{-\alpha}\wedge e_\alpha. \tag{2.6}$$

If $\sigma = \omega_{\mu,J}$, $\mu \neq \mathrm{id}$, then

$$\begin{aligned}\widetilde{r_0} = \frac{t}{2}\Big(&\sum_{\alpha,\beta\in\Delta} s_{\alpha,\beta}\, h_\alpha\wedge h_\beta + 2\sum_{\alpha\in\Phi^+}(e_{-\alpha}\wedge e_\alpha + (-1)^{\chi_{\bar{J}}(\alpha)}e_{-\mu(\alpha)}\wedge e_\alpha)\\ &+\sum_{\alpha\prec\beta}\Big(-\overline{d_{\alpha,\beta}}\, e_\alpha\wedge e_{-\beta} + d_{\alpha,\beta}\left(e_{-\alpha}\wedge e_\beta + (-1)^{\chi_{\bar{J}}(\alpha)}\, e_{-\mu(\alpha)}\wedge e_\beta + (-1)^{\chi_{\bar{J}}(\beta)}\, e_{-\alpha}\wedge e_{\mu(\beta)}\right)\Big)\Big).\end{aligned} \tag{2.7}$$

Proof. We probe simultaneously (2.4) and (2.5). For what follows we use the expressions of r_0 and $(\theta\otimes\theta)(r_0)$ given by (1.36) and (1.37), respectively. Assume that $\sigma = \varsigma_\mu$ with μ arbitrary. In this case, $t\in\mathbb{R}$, so

$$r_0 + (\theta\otimes\theta)(r_0) = \frac{t}{2}\left(\sum_{\alpha,\beta\in\Delta} 2Re(\lambda_{\alpha,\beta})h_\alpha\wedge h_\beta + \sum_{\alpha,\beta\in\Phi^+,\alpha\prec\beta}(d_{\alpha,\beta}\, e_{-\alpha}\wedge e_\beta + \overline{d_{\alpha,\beta}}\, e_\alpha \wedge e_{-\beta})\right)$$

Now,

$$\begin{aligned}(\theta\otimes\mathrm{id} + \mathrm{id}\otimes\theta)(r_0) = \;& \frac{t}{2}\Big(\sum_{\alpha,\beta\in\Delta}\lambda_{\alpha,\beta}(\theta(h_\alpha)\wedge h_\beta + h_\alpha\wedge\theta(h_\beta))\\ &+\sum_{\alpha\in\Phi^+}(\theta(e_{-\alpha})\wedge e_\alpha + e_{-\alpha}\wedge\theta(e_\alpha))\\ &+\sum_{\alpha,\beta\in\Phi^+,\alpha\prec\beta} d_{\alpha,\beta}\left(\theta(e_{-\alpha})\wedge e_\beta + e_{-\alpha}\wedge\theta(e_\beta)\right)\Big)\\ =&-\frac{t}{2}\Big(\sum_{\alpha,\beta\in\Delta}\lambda_{\alpha,\beta}(h_{\mu(\alpha)}\wedge h_\beta + h_\alpha\wedge h_{\mu(\beta)}) + \sum_{\alpha\in\Phi^+}(e_{\mu(\alpha)}\wedge e_\alpha + e_{-\alpha}\wedge e_{-\mu(\alpha)})\\ &+\sum_{\alpha,\beta\in\Phi^+,\alpha\prec\beta} d_{\alpha,\beta}(e_{\mu(\alpha)}\wedge e_\beta + e_{-\alpha}\wedge e_{-\mu(\beta)})\Big).\end{aligned}$$

Here the first term is

$$\sum_{\alpha,\beta\in\Delta}\lambda_{\alpha,\beta}(h_{\mu(\alpha)}\wedge h_\beta+h_\alpha\wedge h_{\mu(\beta)})=\sum_{\alpha,\beta\in\Delta}(\lambda_{\mu(\alpha),\beta}+\lambda_{\alpha,\mu(\beta)})h_\alpha\wedge h_\beta=\sum_{\alpha,\beta\in\Delta}2Re(\lambda_{\alpha,\mu(\beta)})h_\alpha\wedge h_\beta;$$

the first equality by a change of variables, the second because in this case $\lambda_{\alpha,\beta}=\overline{\lambda_{\mu(\alpha),\mu(\beta)}}$, thus $\lambda_{\mu(\alpha),\beta}=\overline{\lambda_{\alpha,\mu(\beta)}}$, and thus $\lambda_{\mu(\alpha),\beta}+\lambda_{\alpha,\mu(\beta)}=2Re(\lambda_{\alpha,\mu\beta})$.

On the other hand, the second term is $\sum_{\alpha\in\Phi^+}e_{\mu\alpha}\wedge e_\alpha=\sum_{\alpha\in\Phi^+,\alpha\neq\mu(\alpha)}e_{\mu(\alpha)}\wedge e_\alpha$. We can enumerate the set of those roots such that $\alpha\neq\mu(\alpha)$ in this way: $\alpha_1,\mu(\alpha_1),\dots,\alpha_k,\mu(\alpha_k)$, where $\alpha_i\neq\mu(\alpha_i)$, $\alpha_i\neq\alpha_j,\mu(\alpha_j)$, for all $i\neq j$. Then $\sum_{\alpha\in\Phi^+}e_{\mu(\alpha)}\wedge e_\alpha=\sum_i e_{\mu(\alpha_i)}\wedge e_{\alpha_i}+e_{\alpha_i}\wedge e_{\mu(\alpha_i)}=0$. In analogous way, $\sum_{\alpha\in\Phi^+}e_{-\alpha}\wedge e_{-\mu(\alpha)}=0$. So,

$$(\theta\otimes\mathrm{id}+\mathrm{id}\otimes\theta)(r_0)=-\frac{t}{2}\Big(\sum_{\alpha,\beta\in\Delta}2Re(\lambda_{\alpha,\mu(\beta)})h_\alpha\wedge h_\beta+\sum_{\alpha,\beta\in\Phi^+,\alpha\prec\beta}d_{\alpha,\beta}(e_{\mu(\alpha)}\wedge e_\beta+e_{-\alpha}\wedge e_{-\mu(\beta)})\Big).$$

Hence $\widetilde{r_0}=r_0+(\theta\otimes\theta)(r_0)-(\mathrm{id}\otimes\theta+\theta\otimes\mathrm{id})(r_0)$ equals the right-hand side of (2.5); when $\mu=\mathrm{id}$, this reduces to (2.4), by Proposition 1.8 (b) (i).

Now, we probe (2.6) and (2.7). Assume that $\sigma=\omega_{\mu,J}$ with μ arbitrary. In this case $t\in i\mathbb{R}$, so

$$\begin{aligned}r_0+(\theta\otimes\theta)r_0=\frac{t}{2}\Big(2i\sum_{\alpha,\beta\in\Delta}Im\lambda_{\alpha,\beta}\,h_\alpha\wedge h_\beta+2\sum_{\alpha\in\Phi^+}e_{-\alpha}\wedge e_\alpha&\\ +\sum_{\alpha,\beta\in\Phi^+,\alpha\prec\beta}(d_{\alpha,\beta}\,e_{-\alpha}\wedge e_\beta-\overline{d_{\alpha,\beta}}\,e_\alpha\wedge e_{-\beta})\Big).&\end{aligned}$$

Now,

$$\begin{aligned}(\theta\otimes\mathrm{id}+\mathrm{id}\otimes\theta)r_0=&\frac{t}{2}\Big(\sum_{\alpha,\beta\in\Delta}\lambda_{\alpha,\beta}(\theta(h_\alpha)\wedge h_\beta+h_\alpha\wedge\theta(h_\beta))+\sum_{\alpha\in\Phi^+}(\theta(e_{-\alpha})\wedge e_\alpha+e_{-\alpha}\wedge\theta(e_\alpha))\\&+\sum_{\alpha,\beta\in\Phi^+,\alpha\prec\beta}d_{\alpha,\beta}(\theta(e_{-\alpha})\wedge e_\beta+e_{-\alpha}\wedge\theta(e_\beta))\Big)\\=&\frac{t}{2}\Big(\sum_{\alpha,\beta\in\Delta}\lambda_{\alpha,\beta}(h_{\mu(\alpha)}\wedge h_\beta+h_\alpha\wedge h_{\mu(\beta)})\\&-\sum_{\alpha\in\Phi^+}(-1)^{\chi_{\bar J}(\alpha)}(e_{-\mu(\alpha)}\wedge e_\alpha+e_{-\alpha}\wedge e_{\mu(\alpha)})\\&-\sum_{\alpha,\beta\in\Phi^+,\alpha\prec\beta}d_{\alpha,\beta}\,((-1)^{\chi_{\bar J}(\alpha)}\,e_{-\mu(\alpha)}\wedge e_\beta+(-1)^{\chi_{\bar J}(\beta)}\,e_{-\alpha}\wedge e_{\mu(\beta)})\Big).\end{aligned}$$

In this case $\lambda_{\alpha,\beta}=-\overline{\lambda_{\mu(\alpha),\mu(\beta)}}$, then $\lambda_{\mu(\alpha),\beta}=-\overline{\lambda_{\alpha,\mu(\beta)}}$, thus $\lambda_{\mu(\alpha),\beta}+\lambda_{\alpha,\mu(\beta)}=2\,i\,Im(\lambda_{\alpha,\mu(\beta)})$ and

$$\sum_{\alpha,\beta\in\Delta}\lambda_{\alpha,\beta}(h_{\mu(\alpha)}\wedge h_\beta+h_\alpha\wedge h_{\mu(\beta)})=\sum_{\alpha,\beta\in\Delta}2\,i\,Im(\lambda_{\alpha,\mu\beta})h_\alpha\wedge h_\beta.$$

Now the second term is

$$\sum_{\alpha\in\Phi^+}(-1)^{\chi_{\bar{J}}(\alpha)}(e_{-\mu(\alpha)}\wedge e_\alpha + e_{-\alpha}\wedge e_{\mu(\alpha)}) = 2\sum_{\alpha\in\Phi^+}(-1)^{\chi_{\bar{J}}(\alpha)}\, e_{-\mu(\alpha)}\wedge e_\alpha.$$

Thus

$$\begin{aligned}(\theta\otimes\mathrm{id}+\mathrm{id}\otimes\theta)(r_0) = \frac{t}{2}\Big(2i\sum_{\alpha,\beta\in\Delta} Im(\lambda_{\alpha,\mu\beta})h_\alpha\wedge h_\beta - 2\sum_{\alpha\in\Phi^+}(-1)^{\chi_{\bar{J}}(\alpha)}\, e_{-\mu(\alpha)}\wedge e_\alpha \\ - \sum_{\alpha,\beta\in\Phi^+,\alpha\prec\beta} d_{\alpha,\beta}\,((-1)^{\chi_{\bar{J}}(\alpha)}\, e_{-\mu(\alpha)}\wedge e_\beta + (-1)^{\chi_{\bar{J}}(\beta)}\, e_{-\alpha}\wedge e_{\mu(\beta)})\Big).\end{aligned}$$

Hence $\widetilde{r_0} = r_0 + (\theta\otimes\theta)(r_0) - \big((\mathrm{id}\otimes\theta)+(\theta\otimes\mathrm{id})\big)(r_0)$ equals the right-hand side of (2.7). When $\mu=\mathrm{id}$, this reduces to (2.6); indeed recall that $\Gamma_1=\Gamma_2=\emptyset$ in this case by [AJ]. $\square$

2.2. **Case $\sigma=\varsigma_\mu$, $\Gamma_1=\Gamma_2=\emptyset$.** We begin with arbitrary μ. Recall the definition of $t_{\alpha,\beta}$ in (2.1).

Proposition 2.3. *$\mathfrak{k}_0$ is a coideal of $\mathfrak{g}_0$ if and only if $t_{\alpha,\beta}=0$ for all $\alpha,\beta\in\Delta$.*

Proof. Recall that in this case $\mathfrak{k}=\langle h_\alpha - h_{\mu(\alpha)},\ e_\alpha - e_{-\mu(\alpha)} : \alpha\in\Phi\rangle$. Here $\widetilde{r_0} = \frac{t}{2}\sum_{\alpha,\beta\in\Delta} t_{\alpha,\beta}h_\alpha\wedge h_\beta$ by (2.4) or (2.5). Note that $t_{\beta,\alpha}=-t_{\alpha,\beta}$. In particular $\mathrm{ad}\,\mathfrak{h}(\widetilde{r_0})=0$. Thus, we only must calculate $\mathrm{ad}(e_\gamma - e_{-\mu(\gamma)})\widetilde{r_0}$, for all $\gamma\in\Phi$. Now:

$$\begin{aligned}-\mathrm{ad}\, e_\gamma\Big(\sum_{\alpha,\beta\in\Delta} t_{\alpha,\beta}h_\alpha\wedge h_\beta\Big) &= \sum_{\alpha,\beta\in\Delta} t_{\alpha,\beta}\big(B(\gamma,\alpha)\, e_\gamma\wedge h_\beta + B(\gamma,\beta)h_\alpha\ \wedge e_\gamma\big)\\ &= \sum_{\alpha,\beta\in\Delta} t_{\alpha,\beta}B(\gamma,\alpha)e_\gamma\wedge h_\beta + \sum_{\alpha,\beta\in\Delta} t_{\beta,\alpha}B(\gamma,\alpha)h_\beta\wedge e_\gamma\\ &= \sum_{\alpha,\beta\in\Delta} t_{\alpha,\beta}B(\gamma,\alpha)e_\gamma\wedge h_\beta - \sum_{\alpha,\beta\in\Delta} t_{\alpha,\beta}B(\gamma,\alpha)h_\beta\wedge e_\gamma\\ &= \sum_{\alpha,\beta\in\Delta} 2t_{\alpha,\beta}B(\gamma,\alpha)e_\gamma\wedge h_\beta = e_\gamma\wedge\sum_{\beta\in\Delta}B(\gamma,\sum_{\alpha\in\Delta}2t_{\alpha,\beta}\alpha)h_\beta.\end{aligned}$$

So

$$\mathrm{ad}(e_\gamma - e_{-\mu(\gamma)})(\widetilde{r_0}) = -\frac{t}{2}e_\gamma\wedge\sum_{\beta\in\Delta}B(\gamma,\sum_{\alpha\in\Delta}2t_{\alpha,\beta}\alpha)h_\beta + \frac{t}{2}e_{-\mu(\gamma)}\wedge\sum_{\beta\in\Delta}B(-\mu(\gamma),\sum_{\alpha\in\Delta}2t_{\alpha,\beta}\alpha)h_\beta.$$

The terms in the right-hand side are linearly independent, so $[e_\gamma - e_{-\mu(\gamma)},\widetilde{r_0}]=0$ if and only if $B(\gamma,\sum_{\alpha\in\Delta}2t_{\alpha,\beta}\alpha)=0$ for all $\gamma\in\Phi^+$, $\beta\in\Delta$. Because of the non-degeneracy of $B(\ ,\ )$, $\mathrm{ad}\,\mathfrak{k}\,\widetilde{r_0}=0$ if and only if $\sum_{\alpha\in\Delta}2t_{\alpha,\beta}\alpha=0$ for all $\beta\in\Delta$, if and only if $t_{\alpha,\beta}=0$ for all $\alpha,\beta\in\Delta$. $\square$

The following propositions follow immediately from the previous one and [AJ, Tables 1.1 and 2.1], see Proposition 1.8.

Proposition 2.4. *If $\sigma = \varsigma$ and $\Gamma_1 = \Gamma_2 = \emptyset$, then $\mathfrak{k}_0$ is a coideal of $\mathfrak{g}_0$ if and only if $\lambda_{\alpha,\beta} = 0$ for all $\alpha, \beta \in \Delta$.* □

Proposition 2.5. *If $\sigma = \varsigma_\mu$ with $\mu \neq \mathrm{id}$ and $\Gamma_1 = \Gamma_2 = \emptyset$, then $\mathfrak{k}_0$ is a coideal of $\mathfrak{g}_0$ if and only if the continuous parameter λ satisfies $\lambda_{\alpha,\beta} = \overline{\lambda_{\mu(\alpha),\mu(\beta)}}$ and $t_{\alpha,\beta} = 0$, for all $\alpha, \beta \in \Delta$.* □

2.3. **Case $\sigma = \varsigma_\mu$, $\Gamma_1 \neq \emptyset$, $\Gamma_2 \neq \emptyset$.** For arbitrary μ, we have:

Proposition 2.6. *$\mathfrak{k}_0$ is not a coideal of $\mathfrak{g}_0$.*

Proof. Recall that $\mathfrak{k}$ is generated by $h_\gamma - h_{\mu(\gamma)}$, $e_\gamma - e_{-\mu(\gamma)}$, $\gamma \in \Phi$, see Lemma 1.4. We study the action of $h_\gamma - h_{\mu(\gamma)}$, $\gamma \in \Phi$, on $\widetilde{r_0}$:

$$\begin{aligned}\mathrm{ad}(h_\gamma - h_{\mu(\gamma)})(\widetilde{r_0}) = \frac{t}{2}\Big(\sum_{\alpha\prec\beta} &(B(\gamma-\mu(\gamma), -\alpha+\beta)\,\overline{d_{\alpha,\beta}}\, e_{-\alpha}\wedge e_\beta + B(\gamma-\mu(\gamma), \alpha-\beta) d_{\alpha,\beta} e_\alpha \wedge e_{-\beta}\\ &+B(\gamma-\mu(\gamma), \mu(\alpha)+\beta)\, d_{\alpha,\beta}\, e_{\mu\alpha}\wedge e_\beta + B(\gamma-\mu(\gamma), -\alpha-\mu(\beta))\, d_{\alpha,\beta}\, e_{-\alpha}\wedge e_{-\mu(\beta)})\Big)\\ &= \frac{t}{2}\sum_{\alpha\prec\beta} B(\gamma-\mu(\gamma), -\alpha+\beta)\, A_{\alpha,\beta},\end{aligned}$$

where $A_{\alpha,\beta} = \overline{d_{\alpha,\beta}} e_{-\alpha}\wedge e_\beta - d_{\alpha,\beta}\, e_\alpha \wedge e_{-\beta} + d_{\alpha,\beta}\, e_{\mu(\alpha)}\wedge e_\beta + d_{\alpha,\beta}\, e_{-\alpha}\wedge e_{-\mu(\beta)}$. Here in the second equality we have argued as follows: $B(\alpha+\mu(\alpha), \gamma-\mu(\gamma)) = 0$, because $\mu^2 = \mathrm{id}$ and μ is B-invariant. Then $B(\gamma-\mu(\gamma), \mu(\alpha)+\beta) = B(\gamma-\mu(\gamma), -\alpha+\beta)$. Also, $B(\gamma-\mu(\gamma), -\alpha-\mu(\beta)) = B(\gamma-\mu(\gamma), -\alpha+\beta)$.

It is clear that the elements $A_{\alpha,\beta}$ are linearly independent, *e. g.* projecting them to $\mathfrak{g}^+\otimes\mathfrak{g}^-$ along the root space decomposition. Thus, $[h_\gamma - h_{\mu(\gamma)}, \widetilde{r_0}] = 0$ if and only if $B(\gamma-\mu(\gamma), \alpha-\beta) = 0$, for all $\gamma\in\Phi, \alpha\in\widehat{\Gamma}_1, \beta\in\widehat{\Gamma}_2, \alpha\prec\beta$. Now, $B(\gamma-\mu(\gamma), \alpha-\beta) = 0$ if and only if $B(\gamma, \alpha-\beta) = B(\mu(\gamma), \alpha-\beta) = B(\gamma, \mu(\alpha-\beta))$, for all $\gamma\in\Phi$ if and only if $\alpha-\beta = \mu(\alpha-\beta)$ if and only if $\alpha+\mu(\beta) = \mu(\alpha)+\beta$. As $\alpha\prec\beta$, we have that $\alpha\neq\beta$. If $\alpha\in\Gamma_1$ and $\beta\in\Gamma_2$, then $\alpha, \mu(\alpha), \beta, \mu(\beta)\in\Delta$ and $\alpha+\mu(\beta) = \mu(\alpha)+\beta$ if and only if $\alpha = \mu(\alpha)$ and $\beta = \mu(\beta)$. We conclude that

$$\mathrm{ad}(h_\gamma - h_{\mu(\gamma)})(\widetilde{r_0}) = 0,\ \forall\gamma\in\Phi \quad \text{if and only if} \quad \alpha = \mu(\alpha),\ \forall\alpha\in\Gamma_1\cup\Gamma_2. \tag{2.8}$$

In view of this, we shall assume in the rest of the proof that $\alpha = \mu(\alpha)$, $\forall\alpha\in\Gamma_1\cup\Gamma_2$. Then,

$$\widetilde{r_0} = \frac{t}{2}\sum_{\alpha,\beta\in\Delta} t_{\alpha,\beta} h_\alpha\wedge h_\beta + \frac{t}{2}\sum_{\alpha\prec\beta} \overline{d_{\alpha,\beta}}\, e_{-\alpha}\wedge e_\beta + d_{\alpha,\beta}\, e_\alpha\wedge e_{-\beta} + d_{\alpha,\beta}\, e_\alpha\wedge e_\beta + d_{\alpha,\beta}\, d_\beta e_{-\alpha}\wedge e_{-\beta}. \tag{2.9}$$

Now, reorder the simple roots in the following way: let $k = \#(\Gamma_1\cup\Gamma_2)$, and let $\alpha_{k+1}, \dots, \alpha_n$ the simple roots that not belong to $\Gamma_1\cup\Gamma_2$. Let $\{\beta_1, \dots, \beta_t\}$ the roots in Γ_1 that are not image of T, then

$$(\alpha_1, \dots, \alpha_k) = (\beta_1, T(\beta_1), \dots, T^{s_1}(\beta_1), \beta_2, \dots, T^{s_2}(\beta_2), \dots, \beta_t, \dots, T^{s_t}(\beta_t)), \tag{2.10}$$

We define $\mathfrak{l}_0 = \sum_{\alpha\in\Delta}\mathbb{R}h_\alpha$, and consider the weight spaces of $\Lambda^2(\mathfrak{g})$ with the order given by Δ. Let $\gamma\in\Delta$ and define $\Gamma(\gamma) = \{\alpha\in\widehat{\Gamma}_1 : \alpha+\gamma\in\Phi\}$. Take $\gamma\in\Delta, \gamma\notin\Gamma_1$ such that $\Gamma(\gamma)\neq\emptyset$ (it is clear

that such γ exists, otherwise $\mathfrak{g}$ would be of type A_1 and in this case $\Gamma_1 = \Gamma_2 = \emptyset$). Take γ_1 in $\Gamma(\gamma)$ and $\gamma_2 \in \widehat{\Gamma}_2$ such that $\gamma_1 \prec \gamma_2$. We will prove that the weight $\gamma_1 + \gamma_2 + \gamma$ occurs in $[e_\gamma - d_\gamma e_{-\mu\gamma}, \widetilde{r_0}]$, so the last one is not zero. Now,

$$[e_\gamma, e_{\gamma_1} \wedge e_{\gamma_2}] = N_{\gamma,\gamma_1} e_{\gamma+\gamma_1} \wedge e_{\gamma_2} + N_{\gamma,\gamma_2} e_{\gamma_1} \wedge e_{\gamma+\gamma_2}, \tag{2.11}$$

and, at least, the first term is not 0, because $\gamma_1 \in \Gamma(\gamma)$. We will see that the term $e_{\gamma+\gamma_1} \wedge e_{\gamma_2}$ is not cancelled by the other terms of $[e_\gamma - d_\gamma e_{-\mu\gamma}, \widetilde{r_0}]$, that are of the form:

$$[e_\gamma, e_{\gamma_1'} \wedge e_{\gamma_2'}] = N_{\gamma,\gamma_1'} e_{\gamma+\gamma_1'} \wedge e_{\gamma_2'} + N_{\gamma,\gamma_2'} e_{\gamma_1'} \wedge e_{\gamma+\gamma_2'}, \tag{2.12}$$

$$[e_\gamma, e_{-\gamma_1'} \wedge e_{\gamma_2'}] = N_{\gamma,-\gamma_1'} e_{\gamma-\gamma_1'} \wedge e_{\gamma_2'} + N_{\gamma,\gamma_2'} e_{-\gamma_1'} \wedge e_{\gamma+\gamma_2'}, \tag{2.13}$$

$$[e_\gamma, e_{\gamma_1'} \wedge e_{-\gamma_2'}] = N_{\gamma,\gamma_1'} e_{\gamma+\gamma_1'} \wedge e_{-\gamma_2'} + N_{\gamma,-\gamma_2'} e_{\gamma_1'} \wedge e_{\gamma-\gamma_2'}, \tag{2.14}$$

$$[e_\gamma, e_{-\gamma_1'} \wedge e_{-\gamma_2'}] = N_{\gamma,-\gamma_1'} e_{\gamma-\gamma_1'} \wedge e_{-\gamma_2'} + N_{\gamma,-\gamma_2'} e_{-\gamma_1'} \wedge e_{\gamma-\gamma_2'}, \tag{2.15}$$

$$[e_{-\mu(\gamma)}, e_{\gamma_1'} \wedge e_{\gamma_2'}] = N_{-\mu(\gamma),\gamma_1'} e_{-\mu(\gamma)+\gamma_1'} \wedge e_{\gamma_2'} + N_{-\mu(\gamma),\gamma_2'} e_{\gamma_1'} \wedge e_{-\mu(\gamma)+\gamma_2'}, \tag{2.16}$$

$$[e_{-\mu(\gamma)}, e_{-\gamma_1'} \wedge e_{\gamma_2'}] = N_{-\mu(\gamma),-\gamma_1'} e_{-\mu(\gamma)-\gamma_1'} \wedge e_{\gamma_2'} + N_{-\mu(\gamma),\gamma_2'} e_{-\gamma_1'} \wedge e_{-\mu(\gamma)+\gamma_2'}, \tag{2.17}$$

$$[e_{-\mu(\gamma)}, e_{\gamma_1'} \wedge e_{-\gamma_2'}] = N_{-\mu(\gamma),\gamma_1'} e_{-\mu(\gamma)+\gamma_1'} \wedge e_{-\gamma_2'} + N_{-\mu(\gamma),-\gamma_2'} e_{\gamma_1'} \wedge e_{-\mu(\gamma)-\gamma_2'}, \tag{2.18}$$

$$[e_{-\mu(\gamma)}, e_{-\gamma_1'} \wedge e_{-\gamma_2'}] = N_{-\mu(\gamma),-\gamma_1'} e_{-\mu(\gamma)-\gamma_1'} \wedge e_{-\gamma_2'} + N_{-\mu(\gamma),-\gamma_2'} e_{-\gamma_1'} \wedge e_{-\mu(\gamma)-\gamma_2'}, \tag{2.19}$$

It is clear that $e_{\gamma+\gamma_1} \wedge e_{\gamma_2}$ cannot be cancelled with terms of type (2.13), (2.14), (2.15), (2.17), (2.18) and (2.19) because of there are negative roots in the factors of the terms of the RHS of these equations (recall that γ and $\mu(\gamma)$ are simple).

It could be cancelation between a term of (2.12) and $e_{\gamma+\gamma_1} \wedge e_{\gamma_2}$ if

(a) $\gamma_1 + \gamma = \gamma_1' + \gamma$ and $\gamma_2 = \gamma_2'$, or
(b) $\gamma_1 + \gamma = \gamma_2'$ and $\gamma_2 = \gamma_1' + \gamma$, or
(c) $\gamma_1 + \gamma = \gamma_1'$ and $\gamma_2 = \gamma_2' + \gamma$, or
(d) $\gamma_1 + \gamma = \gamma_2' + \gamma$ and $\gamma_2 = \gamma_1'$.

Now, (a) could be satisfied if and only if $\gamma_1 = \gamma_1'$ and $\gamma_2 = \gamma_2'$. In the case (b) we have that $\gamma_1 + \gamma = \gamma_2'$, thus $\gamma \in \Gamma_2$. As $\gamma_1 \prec \gamma_2$ and $\gamma_1' \prec \gamma_2'$, there exist $k, s \in \mathbb{N}$ such that $T^k(\gamma_1) = \gamma_2 = \gamma_1' + \gamma$ and $T^s(\gamma_1') = \gamma_2' = \gamma_1 + \gamma$. If we think in terms of the base (2.10) it is easy to see that the last two equalities can not happen simultaneously. The case (c) results in a contradiction because $\gamma_1 + \gamma = \gamma_1'$ implies that γ belongs to Γ_1. Finally, in the case (d) we have that $T^k(\gamma_1) = \gamma_2 = \gamma_1'$ and $T^s(\gamma_1') = \gamma_2' = \gamma_1$, thus $T^{k+s}(\gamma_1) = \gamma_1$, which contradicts the nilpotency of T.

Now, it could be cancellation between a term of (2.16) and $e_{\gamma+\gamma_1} \wedge e_{\gamma_2}$ if

(a) $\gamma_1 + \gamma = \gamma_1' - \mu(\gamma)$ and $\gamma_2 = \gamma_2'$, or
(b) $\gamma_1 + \gamma = \gamma_2'$ and $\gamma_2 = \gamma_1' - \mu(\gamma)$, or
(c) $\gamma_1 + \gamma = \gamma_1'$ and $\gamma_2 = \gamma_2' - \mu(\gamma)$, or
(d) $\gamma_1 + \gamma = \gamma_2' - \mu(\gamma)$ and $\gamma_2 = \gamma_1'$.

In the case (a), as γ and $\mu(\gamma)$ not belongs to Γ_1, we have that $\gamma_1' - \mu(\gamma)$ is not a root, a contradiction because $\gamma_1' - \mu(\gamma) = \gamma_1 + \gamma$. In (b), as $\gamma_1' - \mu(\gamma) = \gamma_2$ is a root, we have that $\mu(\gamma)$ is in Γ_1, thus γ is in Γ_1, a contradiction. In (c), $\gamma_1 + \gamma = \gamma_1'$ implies that $\gamma \in \Gamma_1$, a contradiction. In (d), if $\gamma \neq \mu(\gamma)$, and in consequence $\gamma, \mu(\gamma) \notin \Gamma_1 \cup \Gamma_2$, we have that $\gamma_1 + \gamma + \mu(\gamma) = \gamma_2'$, thus $\gamma \in \Gamma_2$, a contradiction. Thus $\gamma = \mu(\gamma)$ and we have $\gamma_1 + \gamma = \gamma_2' - \gamma$ and $\gamma_2 = \gamma_1'$. Now, $T^k(\gamma_1) = \gamma_2 = \gamma_1'$ and $T^s(\gamma_1') = \gamma_2'$, so $T^{k+s}(\gamma_1) = \gamma_2' = \gamma_1 + 2\gamma$. If we think in terms of the base (2.10) it is easy to see that the last equality can not happen. □

2.4. **Case $\sigma = \omega_J$.** In this case necessarily $\Gamma_1 = \Gamma_2 = \emptyset$. See Proposition 1.8.

Proposition 2.7. *If $\mathfrak{k}_0$ is coideal of $\mathfrak{g}_0$ then $\#J = \#\Delta - 1$; say $\{\alpha\} = \Delta - J$. Assuming this, $\mathfrak{k}_0$ is coideal of $\mathfrak{g}_0$ if and only if the coefficient of α in the largest root of Φ is 1. This happens if and only if*

1. *$\mathfrak{g}_0$ is of type A_n and $\alpha \in \Delta$ is arbitrary, or*
2. *$\mathfrak{g}_0$ is of type B_n and α is the leftmost extreme of the Dynkin diagram (α is the shortest simple root), or*
3. *$\mathfrak{g}_0$ is of type C_n and α is the rightmost extreme of the Dynkin diagram (α is the longest simple root), or*
4. *$\mathfrak{g}_0$ is of type D_n and α is an extreme of the Dynkin diagram, or*
5. *$\mathfrak{g}_0$ is of type E_6 and α is extreme of the long branch of the Dynkin diagram, or*
6. *$\mathfrak{g}_0$ is of type E_7 and α is the extreme of the long branch of the Dynkin diagram.*

Proof. Let us assume that $\sigma = \omega_J$. Recall from (2.6) that

$$\widetilde{r_0} = t \sum_{\alpha \in \Phi^+} (1 + (-1)^{\chi_{\bar{J}}(\alpha)}) e_{-\alpha} \wedge e_\alpha.$$

Step 1. $[h_\gamma, \widetilde{r_0}] = 0$ *for all* $\gamma \in \Delta$.

This is evident. Next, we compute:

Step 2. *If $\gamma \in \Phi^+$, then*

$$\begin{aligned} \operatorname{ad} e_\gamma(\widetilde{r_0}) = t\Big(& (1 + (-1)^{\chi_{\bar{J}}(\gamma)}) h_\gamma \wedge e_\gamma + \sum_{\alpha \in \Phi^+, \gamma - \alpha \in \Phi^+} (1 + (-1)^{\chi_{\bar{J}}(\alpha)}) N_{\gamma, -\alpha}\, e_{\gamma - \alpha} \wedge e_\alpha \\ & + (1 + (-1)^{\chi_{\bar{J}}(\gamma)}) \sum_{\alpha \in \Phi^+, \gamma + \alpha \in \Phi^+} N_{\gamma, -\alpha} (-1)^{\chi_{\bar{J}}(\alpha)}\, e_{-\alpha} \wedge e_{\gamma + \alpha} \Big), \end{aligned} \tag{2.20}$$

$$\begin{aligned} \operatorname{ad} e_{-\gamma}(\widetilde{r_0}) = t\Big(& -(1 + (-1)^{\chi_{\bar{J}}(\gamma)}) e_{-\gamma} \wedge h_\gamma + \sum_{\alpha \in \Phi^+, \gamma - \alpha \in \Phi^+} (1 + (-1)^{\chi_{\bar{J}}(\alpha)}) N_{-\gamma, \alpha}\, e_{-\alpha} \wedge e_{\alpha - \gamma} \\ & - (1 + (-1)^{\chi_{\bar{J}}(\gamma)}) \sum_{\alpha \in \Phi^+, \alpha - \gamma \in \Phi^+} N_{-\gamma, \alpha} (-1)^{\chi_{\bar{J}}(\alpha - \gamma)}\, e_{-\alpha} \wedge e_{-\gamma + \alpha} \Big). \end{aligned} \tag{2.21}$$

Proof. Clearly,

$$\operatorname{ad} e_\gamma(\widetilde{r_0}) = t \sum_{\alpha\in\Phi^+,\alpha\neq\gamma} (1+(-1)^{\chi_{\bar{\jmath}}(\alpha)})\big(N_{\gamma,-\alpha} e_{\gamma-\alpha}\wedge e_\alpha + N_{\gamma,\alpha} e_{-\alpha}\wedge e_{\gamma+\alpha}\big) + t(1+(-1)^{\chi_{\bar{\jmath}}(\gamma)})h_\gamma \wedge e_\gamma.$$

Since $\gamma\in\Phi^+$, we have

$$\begin{aligned}
&\sum_{\alpha\in\Phi^+,\alpha\neq\gamma} (1+(-1)^{\chi_{\bar{\jmath}}(\alpha)})\big(N_{\gamma,-\alpha} e_{\gamma-\alpha}\wedge e_\alpha + N_{\gamma,\alpha} e_{-\alpha}\wedge e_{\gamma+\alpha}\big)\\
&\qquad = \sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (1+(-1)^{\chi_{\bar{\jmath}}(\alpha)}) N_{\gamma,-\alpha} e_{\gamma-\alpha}\wedge e_\alpha\\
&\qquad + \sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^-} (1+(-1)^{\chi_{\bar{\jmath}}(\alpha)}) N_{\gamma,-\alpha} e_{\gamma-\alpha}\wedge e_\alpha\\
&\qquad + \sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (1+(-1)^{\chi_{\bar{\jmath}}(\alpha)}) N_{\gamma,\alpha} e_{-\alpha}\wedge e_{\gamma+\alpha}\\
&\qquad = \sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (1+(-1)^{\chi_{\bar{\jmath}}(\alpha)}) N_{\gamma,-\alpha} e_{\gamma-\alpha}\wedge e_\alpha\\
&\qquad + \sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} \big[(1+(-1)^{\chi_{\bar{\jmath}}(\alpha+\gamma)}) N_{\gamma,-\alpha-\gamma} + (1+(-1)^{\chi_{\bar{\jmath}}(\alpha)}) N_{\gamma,\alpha}\big] e_{-\alpha}\wedge e_{\gamma+\alpha}.
\end{aligned}$$

Now, $(-1)^{\chi_{\bar{\jmath}}(\alpha+\gamma)} = -(-1)^{\chi_{\bar{\jmath}}(\alpha)}(-1)^{\chi_{\bar{\jmath}}(\gamma)}$; and $N_{\gamma,\alpha} = N_{-\alpha-\gamma,\gamma} = -N_{\gamma,-\alpha-\gamma}$ by (1.5) and (1.6). Thus the second sum in the last expression equals

$$\sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} N_{\gamma,\alpha}(-1)^{\chi_{\bar{\jmath}}(\alpha)}(1+(-1)^{\chi_{\bar{\jmath}}(\gamma)}) e_{-\alpha}\wedge e_{\gamma+\alpha},$$

and (2.20) follows. The proof of (2.21) is completely analogous. □

Step 3. *$\mathfrak{k}_0$ is a coideal of $\mathfrak{g}_0$ if and only if for any $\gamma\in\Phi^+$ such that $(-1)^{\chi_{\bar{\jmath}}(\gamma)} = -1$, and for any $\alpha,\beta\in\Phi^+$ such that $\gamma=\alpha+\beta$, one has $(-1)^{\chi_{\bar{\jmath}}(\alpha)} = (-1)^{\chi_{\bar{\jmath}}(\beta)} = -1$.*

Proof. By Lemma 2.1 (1.28) and Step 1, $\mathfrak{k}_0$ is a coideal of $\mathfrak{g}_0$ if and only if $\big(1-(-1)^{\chi_{\bar{\jmath}}(\gamma)}\big)\operatorname{ad} e_\gamma(\widetilde{r_0}) = 0$ for all $\gamma\in\Phi$. If $(-1)^{\chi_{\bar{\jmath}}(\gamma)} = 0$, then there is nothing to prove. If $\gamma\in\Phi^+$ and $(-1)^{\chi_{\bar{\jmath}}(\gamma)} = 1$, then by (2.20) we have

$$\big(1-(-1)^{\chi_{\bar{\jmath}}(\gamma)}\big)\operatorname{ad} e_\gamma(\widetilde{r_0}) = 2t \sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (1+(-1)^{\chi_{\bar{\jmath}}(\alpha)}) N_{\gamma,-\alpha}\, e_{\gamma-\alpha}\wedge e_\alpha.$$

Let $\alpha_1,\dots,\alpha_r\in\Phi^+$ a maximal set satisfying $\beta_1 := \gamma-\alpha_1,\dots,\beta_r := \gamma-\alpha_r\in\Phi^+$ and $\alpha_i\neq\beta_j$ for all i,j. Then the last equation is equivalent to:

$$\big(1-(-1)^{\chi_{\bar{\jmath}}(\gamma)}\big)\operatorname{ad} e_\gamma(\widetilde{r_0}) = 2t\sum_{i=1}^{r} (1+(-1)^{\chi_{\bar{\jmath}}(\alpha_i)}) N_{\gamma,-\alpha_i}\, e_{\beta_i}\wedge e_{\alpha_i} + (1+(-1)^{\chi_{\bar{\jmath}}(\beta_i)}) N_{\gamma,-\beta_i}\, e_{\alpha_i}\wedge e_{\beta_i}.$$

From (1.5) and (1.6) we have

$$N_{\gamma,-\alpha_i} = N_{-\beta_i,\gamma} = -N_{\gamma,-\beta_1},$$

thus

$$\begin{aligned}\big(1-(-1)^{\chi_{\bar J}(\gamma)}\big)\operatorname{ad} e_\gamma(\widetilde{r_0}) &= 2t\sum_{i=1}^{r}(1+(-1)^{\chi_{\bar J}(\alpha_i)})N_{\gamma,-\alpha_i}\, e_{\beta_i}\wedge e_{\alpha_i}\\ &\qquad + (1+(-1)^{\chi_{\bar J}(\beta_i)})(-N_{\gamma,-\alpha_i})\, e_{\alpha_i}\wedge e_{\beta_i}\\ &= 2t\sum_{i=1}^{r}(2+(-1)^{\chi_{\bar J}(\alpha_i)}+(-1)^{\chi_{\bar J}(\beta_i)})N_{\gamma,-\alpha_i}\, e_{\beta_i}\wedge e_{\alpha_i}.\end{aligned}$$

Then $\big(1-(-1)^{\chi_{\bar J}(\gamma)}\big)\operatorname{ad} e_\gamma(\widetilde{r_0}) = 0$ if and only if $(-1)^{\chi_{\bar J}(\alpha)} = -1$ for all $\alpha\in\Phi^+$ such that $\gamma-\alpha\in\Phi^+$. This proves the claim. □

Step 4. *If $\mathfrak{k}_0$ is a coideal of $\mathfrak{g}_0$ then $\#J = \#\Delta - 1$.*

Proof. If $J=\Delta$ then $\mathfrak{g}_0$ is compact, contrary to our assumptions. Thus there is at least one element in $\Delta - J$. Assume that there is more than one element in $\Delta - J$. We can then choose $\alpha\neq\beta\in\Delta-J$ such that the minimal path from α to β in the Dynkin diagram contains only points in J. It follows that there exists $\gamma\in\Phi^+$ satisfying

$$\gamma = \alpha + k_1\alpha_1 + \cdots + k_s\alpha_s + \beta,$$

with $\alpha_1,\dots,\alpha_s\in J$ and $\alpha + k_1\alpha_1+\cdots+k_s\alpha_s\in\Phi^+$. Then, by Lemma 1.2 (c):

$$(-1)^{\chi_{\bar J}(\gamma)} = (-1)^{\chi_{\mathbb{Z}J}(\gamma)+\ell(\gamma)+1} = (-1)^{2k_1+\cdots+2k_s+1} = -1,$$

but $\beta\notin J$, contradicting Step 3. □

Step 5. *Assume that $\Delta - J = \{\alpha\}$. If $\gamma\in\Phi^+$, write $\gamma=\sum_{\beta\in\Delta}k_\beta\beta$. Then $\mathfrak{k}_0$ is coideal of $\mathfrak{g}_0$ if and only if the coefficient k_α is 0 or 1 for any $\gamma\in\Phi^+$.*

Proof. Assume that $\mathfrak{k}_0$ is a coideal of $\mathfrak{g}_0$. If for some $\gamma\in\Phi^+$, $k_\alpha\geq 2$ then we can assume that $k_\alpha = 2$ (for some other positive root, say). Computing $(-1)^{\chi_{\bar J}(\gamma)}$ as in the previous step we get a contradiction. Conversely, assume that the coefficient k_α is 0 or 1 for any $\gamma\in\Phi^+$. Note that $(-1)^{\chi_{\bar J}(\gamma)} = -(-1)^{k_\alpha}$. Thus $(-1)^{\chi_{\bar J}(\gamma)}=-1$ if and only if $k_\alpha = 0$. We conclude now from Step 3. □

Recall that the *largest root* of Φ is the highest weight of the adjoint representation of $\mathfrak{g}$.

Step 6. *Assume that $\Delta - J = \{\alpha\}$. Then $\mathfrak{k}_0$ is coideal of $\mathfrak{g}_0$ if and only if the coefficient of α in the largest root of Φ is 1.*

Proof. If $\sum_{\beta\in\Delta}t_\beta\beta$ is the largest root and $\gamma=\sum_{\beta\in\Delta}k_\beta\beta\in\Phi^+$, then $k_\beta\leq t_\beta$. Thus the Step follows immediately from Step 5. □

It remains only to determine the Dynkin diagrams with a simple root whose coefficient in the largest root is 1. This is an easy task, by inspecting the largest root of each system as listed in [Kn], Appendix C. For example, the largest root corresponding to a system of type B_n is $\alpha_1+\sum_{i=2}^n 2\alpha_i$. So, from

Step 6, $\mathfrak{k}_0$ is subcoideal of $\mathfrak{g}_0$ if and only if $\alpha = \alpha_1$. The same argument applies to the other Dynkin diagrams. $\square$

2.5. **Case $\sigma = \omega_{\mu,J}$, $\mu \neq \mathrm{id}$, $\Gamma_1 = \Gamma_2 = \emptyset$.** We shall show that $\mathfrak{k}_0$ is not coideal of $\mathfrak{g}_0$ except for $\mathfrak{g}_0$ of type A_2. We begin by the following reduction; recall the set $\tilde{J}$ defined in Lemma 1.2.

Lemma 2.8. *If $\gamma \in \Phi^+$ then* $\mathrm{ad}(e_\gamma - (-1)^{\chi_{\tilde J}(\gamma)} e_{\mu(\gamma)})(\widetilde{r_0}) \equiv t u_\gamma \mod \mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}$, *where*

$$\begin{aligned} u_\gamma = & \sum_{\alpha \in \Phi^+, \gamma - \alpha \in \Phi^+} N_{\gamma,-\alpha} e_{\gamma-\alpha} \wedge e_\alpha \\ & + \sum_{\alpha \in \Phi^+, \gamma - \alpha \in \Phi^+} (-1)^{\chi_{\tilde J}(\gamma)+1} N_{\mu(\gamma),-\mu(\alpha)} e_{\mu(\gamma-\alpha)} \wedge e_{\mu(\alpha)} \\ & + \sum_{\alpha \in \Phi^+, \gamma - \alpha \in \Phi^+} (-1)^{\chi_{\tilde J}(\alpha)} (N_{\gamma,-\alpha} - \overline{N_{-\gamma,\alpha}}) e_{\gamma-\alpha} \wedge e_{\mu(\alpha)} \\ & + \sum_{\alpha \in \Phi^+, \gamma + \alpha \in \Phi^+} (-1)^{\chi_{\tilde J}(\gamma+\alpha)} (N_{\alpha,\gamma} + \overline{N_{-\alpha,-\gamma}}) e_{-\alpha} \wedge e_{\mu(\gamma+\alpha)} \\ & + \sum_{\alpha \in \Phi^+, \gamma + \alpha \in \Phi^+} (-1)^{\chi_{\tilde J}(\alpha)} (N_{\gamma,\alpha} + \overline{N_{-\gamma,-\alpha}}) e_{-\mu(\alpha)} \wedge e_{\gamma+\alpha} \end{aligned}$$

Proof. For shortness, let "$X \equiv Y$" mean "$X \equiv Y \mod \mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h}$". We compute:

$$\begin{aligned} \frac{1}{t} \mathrm{ad}\, e_\gamma(\widetilde{r_0}) \equiv & \sum_{\alpha \in \Phi^+} \Big([e_\gamma, e_{-\alpha}] \wedge e_\alpha + e_{-\alpha} \wedge [e_\gamma, e_\alpha] + (-1)^{\chi_{\tilde J}(\alpha)} \big([e_\gamma, e_{-\mu(\alpha)}] \wedge e_\alpha + e_{-\mu(\alpha)} \wedge [e_\gamma, e_\alpha] \big) \Big) \\ \equiv & \sum_{\alpha \in \Phi^+, \alpha \neq \gamma} \big(N_{\gamma,-\alpha} e_{\gamma-\alpha} \wedge e_\alpha + N_{\gamma,\alpha} e_{-\alpha} \wedge e_{\gamma+\alpha} \big) \\ & + \sum_{\alpha \in \Phi^+, \alpha \neq \mu(\gamma)} (-1)^{\chi_{\tilde J}(\alpha)} N_{\gamma,-\mu(\alpha)} e_{\gamma-\mu(\alpha)} \wedge e_\alpha \\ & + \sum_{\alpha \in \Phi^+} (-1)^{\chi_{\tilde J}(\alpha)} N_{\gamma,\alpha} e_{-\mu(\alpha)} \wedge e_{\gamma+\alpha} \end{aligned}$$

$$\equiv$$

$$\sum_{\alpha \in \Phi^+, \gamma - \alpha \in \Phi^+} N_{\gamma,-\alpha} e_{\gamma-\alpha} \wedge e_\alpha \tag{A}$$

$$+ \sum_{\alpha \in \Phi^+, \alpha - \mu(\gamma) \in \Phi^+} (-1)^{\chi_{\tilde J}(\alpha)} N_{\gamma,-\mu(\alpha)} e_{\gamma-\mu(\alpha)} \wedge e_\alpha \tag{B}$$

$$+ \sum_{\alpha \in \Phi^+, \mu(\gamma) - \alpha \in \Phi^+} (-1)^{\chi_{\tilde J}(\alpha)} N_{\gamma,-\mu(\alpha)} e_{\gamma-\mu(\alpha)} \wedge e_\alpha \tag{C}$$

$$+ \sum_{\alpha \in \Phi^+, \gamma + \alpha \in \Phi^+} (-1)^{\chi_{\tilde J}(\alpha)} N_{\gamma,\alpha} e_{-\mu(\alpha)} \wedge e_{\gamma+\alpha}. \tag{D}$$

Here in the third congruence we use that

$$\sum_{\alpha\in\Phi^+,\alpha-\gamma\in\Phi^+} N_{\gamma,-\alpha}e_{\gamma-\alpha}\wedge e_\alpha + \sum_{\alpha\in\Phi^+,\alpha\neq\gamma} N_{\gamma,\alpha}e_{-\alpha}\wedge e_{\gamma+\alpha} = 0$$

by (1.5) and (1.6). Changing α by $\mu(\alpha)$, we have

$$\text{(B)}\qquad \sum_{\alpha\in\Phi^+,\alpha-\gamma\in\Phi^+} (-1)^{\chi_{\bar{j}}(\mu(\alpha))} N_{\gamma,-\alpha}e_{\gamma-\alpha}\wedge e_{\mu(\alpha)}$$

$$\text{(C)}\qquad \sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{j}}(\mu(\alpha))} N_{\gamma,-\alpha}e_{\gamma-\alpha}\wedge e_{\mu(\alpha)}.$$

Now,

$$\begin{aligned}\text{(B)}\qquad & \sum_{\alpha\in\Phi^+,\alpha-\gamma\in\Phi^+} (-1)^{\chi_{\bar{j}}(\mu(\alpha))} N_{\gamma,-\alpha}e_{\gamma-\alpha}\wedge e_{\mu(\alpha)} \\ &= \sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{j}}(\mu(\gamma+\alpha))} N_{\gamma,-\gamma-\alpha}e_{-\alpha}\wedge e_{\mu(\gamma+\alpha)} \\ &= \sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{j}}(\mu(\gamma+\alpha))} N_{\alpha,\gamma}e_{-\alpha}\wedge e_{\mu(\gamma+\alpha)}.\end{aligned}$$

The first equality follows from the change of variables α by $\alpha-\gamma$, and the second from (1.6). In analogous way, we have

$$\frac{1}{t}(-1)^{\chi_{\bar{j}}(\gamma)+1}\operatorname{ad} e_{\mu(\gamma)}(\widetilde{r_0}) \equiv$$

$$\text{(E)}\qquad \sum_{\alpha\in\Phi^+,\mu(\gamma)-\alpha\in\Phi^+} (-1)^{\chi_{\bar{j}}(\gamma)+1} N_{\mu(\gamma),-\alpha}e_{\mu(\gamma)-\alpha}\wedge e_\alpha$$

$$\text{(F)}\qquad + \sum_{\alpha\in\Phi^+,\alpha-\gamma\in\Phi^+} (-1)^{\chi_{\bar{j}}(\alpha)+\chi_{\bar{j}}(\gamma)+1} N_{\mu(\gamma),-\mu(\alpha)}e_{\mu(\gamma)-\mu(\alpha)}\wedge e_\alpha$$

$$\text{(G)}\qquad + \sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{j}}(\alpha)+\chi_{\bar{j}}(\gamma)+1} N_{\mu(\gamma),-\mu(\alpha)}e_{\mu(\gamma)-\mu(\alpha)}\wedge e_\alpha$$

$$\text{(H)}\qquad + \sum_{\alpha\in\Phi^+,\mu(\gamma)+\alpha\in\Phi^+} (-1)^{\chi_{\bar{j}}(\alpha)+\chi_{\bar{j}}(\gamma)+1} N_{\mu(\gamma),\alpha}e_{-\mu(\alpha)}\wedge e_{\mu(\gamma)+\alpha}.$$

By (1.17), we have

$$(-1)^{\chi_{\bar{j}}(\alpha)+\chi_{\bar{j}}(\gamma)+1}N_{\mu(\gamma),-\mu(\alpha)} = (-1)^{\chi_{\bar{j}}(-\mu(\gamma))+\chi_{\bar{j}}(\mu(\alpha))+1}N_{\mu(\gamma),-\mu(\alpha)} = (-1)^{\chi_{\bar{j}}(\mu(\gamma-\alpha))+1}\overline{N_{-\gamma,\alpha}}. \tag{2.22}$$

Changing $\gamma-\alpha$ by α and using $x\wedge y = -y\wedge x$, we have

$$\text{(E)}\qquad \sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{j}}(\gamma)+1} N_{\mu(\gamma),-\mu(\alpha)}e_{\mu(\gamma-\alpha)}\wedge e_{\mu(\alpha)}$$

$$\text{(F)}\qquad \sum_{\alpha\in\Phi^-,\gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{j}}(\mu(\alpha))} \overline{N_{-\gamma,\gamma-\alpha}}e_{\gamma-\alpha}\wedge e_{\mu(\alpha)}$$

$$\text{(G)}\qquad \sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{j}}(\mu(\alpha))} \overline{N_{-\gamma,\gamma-\alpha}}e_{\gamma-\alpha}\wedge e_{\mu(\alpha)}$$

Now, performing the change of variable α by $-\alpha$ and using (1.5) and (1.6), we have

$$\text{(F)} \qquad \begin{aligned} &\sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\mu(\alpha))}\overline{N_{-\gamma,\gamma+\alpha}}e_{\gamma+\alpha}\wedge e_{-\mu(\alpha)} \\ &= \sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\mu(\alpha))}\overline{N_{-\alpha,-\gamma}}e_{\gamma+\alpha}\wedge e_{-\mu(\alpha)} \\ &= \sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\mu(\alpha))}\overline{N_{-\gamma,-\alpha}}e_{-\mu(\alpha)}\wedge e_{\gamma+\alpha}. \end{aligned}$$

By (1.6), we have the following expression for (G):

$$\text{(G)} \qquad \sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\mu(\alpha))+1}\overline{N_{-\gamma,\alpha}}e_{\gamma-\alpha}\wedge e_{\mu(\alpha)}$$

For (H), we perform the change of variables α by $\mu(\alpha)$; applying (1.16), (1.5), we get:

$$\text{(H)} \qquad \begin{aligned} &\sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\mu(\alpha))+\chi_{\bar{\jmath}}(\gamma)+1}N_{\mu(\gamma),\mu(\alpha)}e_{-\alpha}\wedge e_{\mu(\gamma+\alpha)} \\ &= \sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(-\mu(\gamma))+\chi_{\bar{\jmath}}(-\mu(\alpha))+1}N_{\mu(\gamma),\mu(\alpha)}e_{-\alpha}\wedge e_{\mu(\gamma+\alpha)} \\ &= \sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\mu(\gamma+\alpha))}\overline{N_{-\alpha,-\gamma}}e_{-\alpha}\wedge e_{\mu(\gamma+\alpha)}. \end{aligned}$$

Finally, (B) + (H) is

$$\sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\mu(\gamma+\alpha))}(N_{\alpha,\gamma}+\overline{N_{-\alpha,-\gamma}})e_{-\alpha}\wedge e_{\mu(\gamma+\alpha)},$$

(C) + (G) is

$$\sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\mu(\alpha))}(N_{\gamma,-\alpha}-\overline{N_{-\gamma,\alpha}})e_{\gamma-\alpha}\wedge e_{\mu(\alpha)},$$

and (D) + (F) is

$$\sum_{\alpha\in\Phi^+,\gamma+\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\alpha)}(N_{\gamma,\alpha}+\overline{N_{-\gamma,-\alpha}})e_{-\mu(\alpha)}\wedge e_{\gamma+\alpha}$$

□

Remark 2.9. If $\gamma\in\Phi^+$ then by the previous Lemma,

$$\mathrm{ad}(e_\gamma-(-1)^{\chi_{\bar{\jmath}}(\gamma)}e_{\mu(\gamma)})(\widetilde{r_0})\equiv tv_\gamma \mod \left(\mathfrak{h}\otimes\mathfrak{g}+\mathfrak{g}\otimes\mathfrak{h}+\mathfrak{n}^+\otimes\mathfrak{n}^-+\mathfrak{n}^-\otimes\mathfrak{n}^+\right),$$

where

$$\begin{aligned} v_\gamma = &\sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} N_{\gamma,-\alpha}e_{\gamma-\alpha}\wedge e_\alpha \\ &+\sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\gamma)+1}N_{\mu(\gamma),-\mu(\alpha)}e_{\mu(\gamma-\alpha)}\wedge e_{\mu(\alpha)} \\ &+\sum_{\alpha\in\Phi^+,\gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{\jmath}}(\alpha)}(N_{\gamma,-\alpha}-\overline{N_{-\gamma,\alpha}})e_{\gamma-\alpha}\wedge e_{\mu(\alpha)}. \end{aligned}$$

It is clear that $v_\gamma \neq 0$ implies $\operatorname{ad}(e_\gamma - (-1)^{\chi_{\tilde J}(\gamma)} e_{\mu(\gamma)})(\widetilde{r_0}) \neq 0$.

Let V a finite dimensional representation of $\mathfrak{g}$. If $\lambda \in \mathfrak{h}^*$, then we denote $V_{(\lambda)} := \{v \in V : h.v = \lambda(h)v\}$. If $v \in V$, then we denote $v_{(\lambda)}$ the component of weight λ of v.

Corollary 2.10. *If $\gamma \in \Phi^+$ then*

$$\operatorname{ad}(e_\gamma - (-1)^{\chi_{\tilde J}(\gamma)} e_{\mu(\gamma)})(\widetilde{r_0})_{(\gamma)} \equiv t w_\gamma \mod \big(\mathfrak{h} \otimes \mathfrak{g} + \mathfrak{g} \otimes \mathfrak{h} + \mathfrak{n}^+ \otimes \mathfrak{n}^- + \mathfrak{n}^- \otimes \mathfrak{n}^+\big),$$

where

$$w_\gamma = 2 \sum_{\alpha \in \Phi^+, \gamma-\alpha \in \Phi^+} N_{\gamma,-\alpha} e_{\gamma-\alpha} \wedge e_\alpha + \sum_{\alpha \in (\Phi^+)^\mu, \gamma-\alpha \in \Phi^+} (-1)^{\chi_{\tilde J}(\alpha)} (N_{\gamma,-\alpha} - \overline{N_{-\gamma,\alpha}}) e_{\gamma-\alpha} \wedge e_\alpha.$$

□

Lemma 2.11. *Let $\sigma = \omega_{\mu,J}$, with $J = \{\alpha_2\}$. Let $\alpha_1, \alpha_3 \in \Delta$ such that $\alpha_1, \alpha_2, \alpha_3$ is a subsystem of type A_3 and $\sigma(\alpha_1) = \alpha_3$. Then $\chi_{\tilde J}(\alpha_1 + \alpha_2 + \alpha_3) = 1$.*

Proof. By the Jacobi identity we have that

$$[e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_3}]] = -[e_{\alpha_2}, [e_{\alpha_3}, e_{\alpha_1}]] - [e_{\alpha_3}, [e_{\alpha_1}, e_{\alpha_2}]] = -[e_{\alpha_3}, [e_{\alpha_1}, e_{\alpha_2}]].$$

Now $[e_{\alpha_1}, [e_{\alpha_2}, e_{\alpha_3}]] = N_{\alpha_2,\alpha_3} N_{\alpha_1,\alpha_2+\alpha_3}$ and $-[e_{\alpha_3}, [e_{\alpha_1}, e_{\alpha_2}]] = -N_{\alpha_1,\alpha_2} N_{\alpha_3,\alpha_1+\alpha_1}$, thus

$$N_{\alpha_2,\alpha_3} N_{\alpha_1,\alpha_2+\alpha_3} = -N_{\alpha_1,\alpha_2} N_{\alpha_3,\alpha_1+\alpha_1}. \tag{2.23}$$

From (1.17), we have that

$$(-1)^{\chi_{\tilde J}(\alpha_2) + \chi_{\tilde J}(\alpha_3)} N_{\alpha_2,\alpha_1} = (-1)^{\chi_{\tilde J}(\alpha_2+\alpha_3)} \overline{N_{-\alpha_2,-\alpha_3}}.$$

As $\chi_{\tilde J}(\alpha_2) = 1$ and $\chi_{\tilde J}(\alpha_3) = 0$, we have

$$N_{\alpha_1,\alpha_2} = -N_{\alpha_2,\alpha_1} = (-1)^{\chi_{\tilde J}(\alpha_2+\alpha_3)} \overline{N_{-\alpha_2,-\alpha_3}}. \tag{2.24}$$

Again from (1.17), we have

$$(-1)^{\chi_{\tilde J}(\alpha_1) + \chi_{\tilde J}(\alpha_2+\alpha_3)} N_{\alpha_3,\alpha_2+\alpha_1} = (-1)^{\chi_{\tilde J}(\alpha_1+\alpha_2+\alpha_3)} \overline{N_{-\alpha_1,-\alpha_2-\alpha_3}}.$$

As $\chi_{\tilde J}(\alpha_1) = 0$, we get

$$(-1)^{\chi_{\tilde J}(\alpha_2+\alpha_3)} N_{\alpha_3,\alpha_2+\alpha_1} = (-1)^{\chi_{\tilde J}(\alpha_1+\alpha_2+\alpha_3)} \overline{N_{-\alpha_1,-\alpha_2-\alpha_3}}. \tag{2.25}$$

We multiply equation (2.25) by N_{α_1,α_2} and obtain

$$(-1)^{\chi_{\tilde J}(\alpha_2+\alpha_3)} N_{\alpha_1,\alpha_2} N_{\alpha_3,\alpha_2+\alpha_1} = (-1)^{\chi_{\tilde J}(\alpha_1+\alpha_2+\alpha_3)} N_{\alpha_1,\alpha_2} \overline{N_{-\alpha_1,-\alpha_2-\alpha_3}},$$

applying (2.23) to the N_{α_1,α_2} on the left, we have

$$-(-1)^{\chi_{\tilde J}(\alpha_2+\alpha_3)} N_{\alpha_2,\alpha_3} N_{\alpha_1,\alpha_2+\alpha_3} = (-1)^{\chi_{\tilde J}(\alpha_1+\alpha_2+\alpha_3)} N_{\alpha_1,\alpha_2} \overline{N_{-\alpha_1,-\alpha_2-\alpha_3}}.$$

Applying (2.24) we obtain

$$-(-1)^{\chi_{\tilde{J}}(\alpha_2+\alpha_3)}N_{\alpha_2,\alpha_3}N_{\alpha_1,\alpha_2+\alpha_3}=(-1)^{\chi_{\tilde{J}}(\alpha_1+\alpha_2+\alpha_3)}(-1)^{\chi_{\tilde{J}}(\alpha_2+\alpha_3)}\overline{N_{-\alpha_2,-\alpha_3}N_{-\alpha_1,-\alpha_2-\alpha_3}},$$

thus

$$-N_{\alpha_2,\alpha_3}N_{\alpha_1,\alpha_2+\alpha_3}=(-1)^{\chi_{\tilde{J}}(\alpha_1+\alpha_2+\alpha_3)}\overline{N_{-\alpha_2,-\alpha_3}N_{-\alpha_1,-\alpha_2-\alpha_3}}.$$

Now, from (1.7) we have that $N_{-\alpha_2,-\alpha_3}=c_1N_{\alpha_2,\alpha_3}$ and $N_{-\alpha_1,-\alpha_2-\alpha_3}=c_2N_{\alpha_1,\alpha_2+\alpha_3}$ with $c_1,c_2<0$, thus

$$-cN_{\alpha_2,\alpha_3}N_{\alpha_1,\alpha_2+\alpha_3}=(-1)^{\chi_{\tilde{J}}(\alpha_1+\alpha_2+\alpha_3)}\overline{N_{\alpha_2,\alpha_3}N_{\alpha_1,\alpha_2+\alpha_3}}$$

with $c>0$. This clearly implies that $\chi_{\tilde{J}}(\alpha_1+\alpha_2+\alpha_3)=1$. □

Proposition 2.12. *Let $\sigma=\omega_{\mu,J}$, $\Gamma_1=\Gamma_2=\emptyset$. If $\mathfrak{g}$ is of type A_2, then $\mathfrak{k}_0$ is coideal of $\mathfrak{g}_0$. In the other cases, $\mathfrak{k}_0$ is not coideal of $\mathfrak{g}_0$.*

Proof. When $\mathfrak{g}$ is of type A_2 is easy to check directly that $\operatorname{ad}(e_\gamma-(-1)^{\chi_{\tilde{J}}(\gamma)}e_{\mu(\gamma)})(\widetilde{r_0})=0$ for all $\gamma\in\Phi^+$.

In the other cases, the Dynkin diagrams that admit non trivial automorphism have subdiagrams of type A_3 or A_4 where μ acts non trivially. In the follows we prove that if we have subdiagrams of type A_3 or A_4 where μ acts non trivially, then there exist $\gamma\in\Phi^+$ such that $v_\gamma\neq 0$.

Type A_3. Let α_1, α_2, α_3 be simple roots such that they determine a subdiagram of type A_3 and μ restricted to $\alpha_1,\alpha_2,\alpha_3$ is non trivial, i.e. $\mu(\alpha_1)=\alpha_3$ and $\mu(\alpha_2)=\alpha_2$. Now we will consider two subcases $J=\emptyset$ and $J\neq\emptyset$.

A_3 **and** $J=\emptyset$. Let $\gamma=\alpha_1+\alpha_2$, thus $\mu(\gamma)=\alpha_2+\alpha_3$. Then

$$\begin{aligned}v_\gamma=\ & N_{\gamma,-\alpha_1}e_{\alpha_2}\wedge e_{\alpha_1}+N_{\gamma,-\alpha_2}e_{\alpha_1}\wedge e_{\alpha_2}\\ &+(-1)^{\chi_{\tilde{J}}(\gamma)+1}N_{\mu(\gamma),-\alpha_3}e_{\alpha_2}\wedge e_{\alpha_3}+(-1)^{\chi_{\tilde{J}}(\gamma)+1}N_{\mu(\gamma),-\alpha_2}e_{\alpha_3}\wedge e_{\alpha_2}\\ &+(-1)^{\chi_{\tilde{J}}(\alpha_1)}(N_{\gamma,-\alpha_1}-\overline{N_{-\gamma,\alpha_1}})e_{\alpha_2}\wedge e_{\alpha_3}+(-1)^{\chi_{\tilde{J}}(\alpha_2)}(N_{\gamma,-\alpha_2}-\overline{N_{-\gamma,\alpha_2}})e_{\alpha_1}\wedge e_{\alpha_2}.\end{aligned}$$

Now, $N_{\gamma,-\alpha_1}=N_{\alpha_1+\alpha_2,-\alpha_1}=N_{-\alpha_1,-\alpha_2}=N_{-\alpha_2,\alpha_1+\alpha_2}=-N_{\gamma,-\alpha_2}$. As $J=\emptyset$, we have $\chi_{\tilde{J}}(\alpha_1)=\chi_{\tilde{J}}(\alpha_2)=0$. Finally, by (1.7), we have

$$\overline{N_{-\gamma,\alpha_2}}=\frac{1}{(-\gamma|\alpha_2)\overline{N_{\gamma,-\alpha_2}}}.$$

Thus,

$$\begin{aligned}v_\gamma=\ & (3N_{\gamma,-\alpha_2}-\frac{1}{(-\gamma|\alpha_2)\overline{N_{\gamma,-\alpha_2}}})e_{\alpha_1}\wedge e_{\alpha_2}+ce_{\alpha_2}\wedge e_{\alpha_3}\\ =&\frac{3N_{\gamma,-\alpha_2}\overline{N_{\gamma,-\alpha_2}}(\gamma|\alpha_2)+1}{(\gamma|\alpha_2)\overline{N_{\gamma,-\alpha_2}}}e_{\alpha_1}\wedge e_{\alpha_2}+ce_{\alpha_2}\wedge e_{\alpha_3}\\ =&\frac{3||N_{\gamma,-\alpha_2}||^2(\gamma|\alpha_2)+1}{(\gamma|\alpha_2)\overline{N_{\gamma,-\alpha_2}}}e_{\alpha_1}\wedge e_{\alpha_2}+ce_{\alpha_2}\wedge e_{\alpha_3}.\end{aligned}$$

As $(\gamma|\alpha_2) > 0$, we have $v_\gamma \neq 0$.

Type A_3 **and** $J \neq \emptyset$. Let $\gamma = \alpha_1+\alpha_2+\alpha_3$, thus $\mu(\gamma) = \gamma$ and from Lemma 2.11 we have $\chi_{\bar{J}}(\gamma) = 1$. From Corollary 2.10 we have that $v_\gamma = w_\gamma$ andFrom Corollary 2.10 we have

$$\begin{aligned} w_\gamma = \quad & 2 \sum_{\alpha\in\Phi^+, \gamma-\alpha\in\Phi^+} N_{\gamma,-\alpha} e_{\gamma-\alpha} \wedge e_\alpha \\ & + \sum_{\alpha\in(\Phi^+)^\mu, \gamma-\alpha\in\Phi^+} (-1)^{\chi_{\bar{J}}(\alpha)} (N_{\gamma,-\alpha} - \overline{N_{-\gamma,\alpha}}) e_{\gamma-\alpha} \wedge e_\alpha \end{aligned}$$

Now there exists no $\alpha \in (\Phi^+)^\mu$ such that $\gamma - \alpha \in \Phi^+$; hence we have

$$\begin{aligned} \frac{1}{2} w_\gamma = & \sum_{\alpha\in\Phi^+, \gamma-\alpha\in\Phi^+,} N_{\gamma,-\alpha} e_{\gamma-\alpha} \wedge e_\alpha \\ = & N_{\gamma,-\alpha_1} e_{\alpha_2+\alpha_3} \wedge e_{\alpha_1} + N_{\gamma,-\alpha_3} e_{\alpha_1+\alpha_2} \wedge e_{\alpha_3} + N_{\gamma,-\alpha_1-\alpha_2} e_{\alpha_3} \wedge e_{\alpha_1+\alpha_2} + N_{\gamma,-\alpha_2-\alpha_3} e_{\alpha_1} \wedge e_{\alpha_2+\alpha_3} \\ = & (N_{\gamma,-\alpha_2-\alpha_3} - N_{\gamma,-\alpha_1}) e_{\alpha_1} \wedge e_{\alpha_2+\alpha_3} + (N_{\gamma,-\alpha_1-\alpha_2} - N_{\gamma,-\alpha_3}) e_{\alpha_3} \wedge e_{\alpha_1+\alpha_2}. \end{aligned}$$

Now $N_{\gamma,-\alpha_2-\alpha_3} = N_{-\alpha_1,\gamma} = -N_{\gamma,-\alpha_1}$ and $N_{\gamma,-\alpha_1-\alpha_2} = N_{-\alpha_3,\gamma} = -N_{\gamma,-\alpha_3}$, thus

$$\frac{1}{2} w_\gamma = 2N_{\gamma,-\alpha_2-\alpha_3} e_{\alpha_1} \wedge e_{\alpha_2+\alpha_3} + 2N_{\gamma,-\alpha_1-\alpha_2} e_{\alpha_3} \wedge e_{\alpha_1+\alpha_2} \neq 0.$$

Type A_4. Let α_1, α_2, α_3, α_4 be simple roots such that they determine a subdiagram of type A_4 where μ restricted to $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ is non trivial, i. e. $\mu(\alpha_1) = \alpha_4$ and $\mu(\alpha_2) = \alpha_3$. Let $\gamma = \alpha_1 + \alpha_2$ and it is clear that $\mu(\gamma) = \alpha_3 + \alpha_4$. Then

$$\begin{aligned} v_\gamma = & N_{\gamma,-\alpha_1} e_{\alpha_2} \wedge e_{\alpha_1} + N_{\gamma,-\alpha_2} e_{\alpha_1} \wedge e_{\alpha_2} \\ & + (-1)^{\chi_{\bar{J}}(\gamma)+1} N_{\mu(\gamma),-\alpha_4} e_{\alpha_3} \wedge e_{\alpha_4} + (-1)^{\chi_{\bar{J}}(\gamma)+1} N_{\mu(\gamma),-\alpha_3} e_{\alpha_4} \wedge e_{\alpha_3} \\ & + (-1)^{\chi_{\bar{J}}(\alpha_1)} (N_{\gamma,-\alpha_1} - \overline{N_{-\gamma,\alpha_1}}) e_{\alpha_2} \wedge e_{\alpha_4} + (-1)^{\chi_{\bar{J}}(\alpha_2)} (N_{\gamma,-\alpha_2} - \overline{N_{-\gamma,\alpha_2}}) e_{\alpha_1} \wedge e_{\alpha_3}. \end{aligned}$$

Now, from (1.5) and (1.6), we have

$$N_{\gamma,-\alpha_1} e_{\alpha_2} \wedge e_{\alpha_1} + N_{\gamma,-\alpha_2} e_{\alpha_1} \wedge e_{\alpha_2} = (N_{\gamma,-\alpha_2} - N_{\gamma,-\alpha_1}) e_{\alpha_1} \wedge e_{\alpha_2} = 2N_{\gamma,-\alpha_2} e_{\alpha_1} \wedge e_{\alpha_2} \neq 0.$$

□

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